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Abstract

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MATHEMATICS

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ON OPTIMAL PROCESSES IN LINEAR SYSTEMS THAT DO NOT SATISFY THE NON-DEGENERACY CONDITION

(Presented by Academician P. S. Aleksandrov on 26 XI 1969)

In his paper ⁽¹⁾ V. G. Boltyanskii established necessary and sufficient conditions of optimality for a linear problem with moving endpoints under the assumption that the linear problem is nondegenerate and that the set M_1 , on which the process terminates, is strongly stable. In the present paper this problem is considered in the case when the set M_1 is stable (and not strongly stable); the nondegeneracy condition introduced in ⁽¹⁾ is also not assumed to hold.

Consider the linear controlled object

$$\dot{x} = Ax + v \quad (1)$$

with constant (real) coefficients, where $x = (x^1, x^2, \dots, x^n)$ is the phase variable, $v = (v^1, v^2, \dots, v^k)$ is the control parameter, or so-called object (A, V) . It is assumed that the control domain V is a closed bounded convex set of the phase space. An admissible control is any piecewise-continuous function $v(t)$ with values in the set V , given on some interval $t_0 \leq t \leq t_1$.

Let M_0 and M_1 be some closed convex sets given in the phase space X ; let $v(t)$, $t_0 \leq t \leq t_1$, be an admissible control, and $x(t)$, $t_0 \leq t \leq t_1$, the corresponding trajectory that effects the transition from the set M_0 to the set M_1 : $x(t_0) \in M_0$, $x(t_1) \in M_1$. Then the number $t_1 - t_0$ is called the **transition time** from M_0 to M_1 along the trajectory $x(t)$, $t_0 \leq t \leq t_1$. The process $(x(t), v(t))$, $t_0 \leq t \leq t_1$, effecting the transition from M_0 to M_1 , is called optimal if the transition time from M_0 to M_1 by means of any other admissible controls is not smaller.

It is useful to introduce the following definitions. Let $(v(t), x(t))$, $t_0 \leq t \leq t_1$, be a process effecting the transition from the set M_0 to the set M_1 . If there exists a nontrivial solution $\psi(t)$ of the auxiliary system

$$\dot{\psi} = -A^*\psi, \quad (2)$$

for which the functions $v(t)$, $x(t)$, $\psi(t)$ satisfy the maximum condition and the transversality conditions (see ⁽²⁾) at the left and right endpoints, then we shall agree to call the process $(v(t), x(t))$, $t_0 \leq t \leq t_1$, extremal. Further, if there exists a nontrivial solution $\psi(t)$ of system (2) for which the extremal process $(v(t), x(t))$, $t_0 \leq t \leq t_1$, also satisfies the additional condition

$$H(t) = \psi(t)\dot{x}(t) \equiv 0,$$

then we shall agree to call such an extremal process exceptional.

Now theorem 1 (⁽¹⁾, p. 785) can be formulated as follows:

Theorem 1. *If the set M_1 is strongly stable, then for optimality of the process $(v(t), x(t))$, $t_0 \leq t \leq t_1$, it is necessary and sufficient that it be extremal.*

Before proceeding to the subject proper of our investigation, let us note two more simple facts that follow directly from work ⁽¹⁾ (the set M_1 is assumed to be stable, not strongly stable).

Proposition 1. *If $(v(t), x(t))$, $t_0 \leq t \leq t_1$, is an extremal process effecting a transition from the set M_0 to the set M_1 , then the function*

$$H(t) = \psi(t)\dot{x}(t) = \psi(t)(Ax(t) + v(t))$$

satisfies the condition $H(t) = \text{const} \geq 0$.

The constancy of the function H was proved in ⁽²⁾. The nonnegativity of this constant follows from the relation

$$H(t_0) = \psi(t_0)\dot{x}(t_0) = n_0 \lim_{t \rightarrow t_0} \frac{x(t) - x(t_0)}{t - t_0} = n_0 \lim_{t \rightarrow t_0} \frac{x(t) - x_0}{t - t_0}$$

and from the fact that $x(t) \in Y_{t_1-t_0}$ for $t_0 \leq t \leq t_1$ (see relation (11) in ⁽¹⁾).

Proposition 2. *If $(v(t), x(t))$, $t_0 \leq t \leq t_1$, is an extremal process effecting a transition from the set M_0 to the set M_1 , then for any θ , $t_0 \leq \theta \leq t_1$, the hyperplane Γ_θ , passing through the point $x(\theta)$ and orthogonal to the vector $\psi(\theta)$, is a supporting hyperplane of the convex set $Y_{t_1-\theta}$, i.e.*

$$\psi(\theta)(x^* - x(\theta)) \geq 0 \quad \text{for } x^* \in Y_{t_1-\theta}. \quad (3)$$

For the proof of Proposition 2 see ⁽¹⁾, p. 793.

Let us observe (see the examples in ⁽¹⁾) that without the assumption of strong stability, extremality is no longer a sufficient condition for optimality. In particular, example 4 of ⁽¹⁾ shows that if the set M_1 is stable but not strongly stable, then an extremal process, even one that is not exceptional, may fail to be optimal.

In other words, the requirement of extremality, even under the additional condition $H > 0$, still does not ensure optimality. It is curious to note that, although

the extremal process indicated in example 4 of ⁽¹⁾ is itself not exceptional, the faster transition process demonstrating the nonoptimality of the process $(v(t), z(t))$ itself already satisfies the condition $H = 0$, i.e. is exceptional. The following theorem shows that this fact is general.

Theorem 2. *If (under the assumption of stability of the set M_1) an extremal process $(v(t), x(t))$, $t_0 \leq t \leq t_1$, effecting a transition from the set M_0 to M_1 , is not optimal, then there exists an exceptional extremal process effecting a transition from M_0 to M_1 in a shorter time.*

Proof. Suppose that the extremal process $(v(t), x(t))$, $t_0 \leq t \leq t_1$, is not optimal. Then there exists an admissible control $\tilde{v}(t)$ such that, under its action, the phase point, starting at the moment t_0 from some position $\tilde{x}_0 \in M_0$, reaches some point of the set M_1 at the time $\tau < t_1$ (i.e. earlier than in the motion along the trajectory $x(t)$). We denote by $\tilde{x}(t)$ the phase trajectory issuing from the point \tilde{x}_0 and corresponding to the control $\tilde{v}(t)$. By assumption, $\tilde{x}(t_0) = \tilde{x}_0 \in M_0$, $\tilde{x}(\tau) \in M_1$.

Let θ be an arbitrary point of the interval $t_0 \leq t \leq \tau$. Since the functions $v(t)$, $x(t)$, $\psi(t)$ satisfy the maximum condition and the transversality condition at the left endpoint, it follows, according to Lemma 4 of work ⁽¹⁾, that we have:

$$\psi(\theta)x(\theta) - \psi(\theta)\tilde{x}(\theta) \geq 0 \quad \text{for } t_0 \leq \theta \leq \tau. \quad (4)$$

* For the definitions of stability, strong stability of the set M_1 , and the attainability sphere of the set M_1 , see ⁽¹⁾.

From the point $\tilde{x}(\theta)$ one can pass to the set M_1 in time $\tau - \theta$ (along the trajectory $\tilde{x}(t)$, $\theta \leq t \leq \tau$), i.e. $\tilde{x}(\theta) \in Y_{\tau-\theta} \subset Y_{t_1-\theta}$. Consequently, by virtue of (3), $\psi(\theta)(\tilde{x}(\theta) - x(\theta)) \geq 0$. Comparing this with inequality (4), we obtain

$$\psi(\theta)(\tilde{x}(\theta) - x(\theta)) \equiv 0 \quad \text{for } t_0 \leq \theta \leq \tau. \quad (5)$$

Let now θ_0 be an arbitrary point of continuity of the controls $v(t)$, $\tilde{v}(t)$, lying in the interval $t_0 \leq t \leq \tau$, such that $x(t)$, $\tilde{x}(t)$ have at the point $t = \theta_0$ a continuous derivative

$$\dot{x}(\theta_0) = Ax(\theta_0) + v(\theta_0), \quad \dot{\tilde{x}}(\theta_0) = A\tilde{x}(\theta_0) + \tilde{v}(\theta_0). \quad (6)$$

Moreover, the function $\psi(\theta)(\tilde{x}(\theta_0) - x(\theta))$ of the variable θ attains a minimum at the point $\theta = \theta_0$, and therefore

$$\left. \frac{d}{d\theta} \psi(\theta)(\tilde{x}(\theta_0) - x(\theta)) \right|_{\theta=\theta_0} = \dot{\psi}(\theta_0)(\tilde{x}(\theta_0) - x(\theta_0)) = \psi(\theta_0)\dot{x}(\theta_0) = 0.$$

Taking (2) and (6) into account, we obtain from this

$$(-A^*\psi(\theta_0))(\tilde{x}(\theta_0) - x(\theta_0)) - \psi(\theta_0)(Ax(\theta_0) + v(\theta_0)) = 0$$

or

$$-\psi(\theta_0)(A\tilde{x}(\theta_0) - Ax(\theta_0)) - \psi(\theta_0)(Ax(\theta_0) + v(\theta_0)) = 0,$$

whence

$$\psi(\theta_0)(A\tilde{x}(\theta_0) + v(\theta_0)) = 0. \quad (7)$$

Differentiating relation (5) for $\theta = \theta_0$, we obtain, by virtue of (2) and (6),

$$\psi(\theta_0)(\tilde{v}(\theta_0) - v(\theta_0)) = 0, \quad (8)$$

and therefore, according to (7),

$$\psi(\theta_0)(A\tilde{x}(\theta_0) + \tilde{v}(\theta_0)) = 0, \quad \text{i.e. } \psi(\theta_0)\dot{\tilde{x}}(\theta_0) = 0. \quad (9)$$

Relations (8) and (9) hold at the points of continuity of the controls $v(t)$ and $\tilde{v}(t)$, i.e. at all points of the segment $[t_0, \tau]$ except for a finite number of points.

Using relations (8), (5), (3), and (9), it is not difficult to prove that the functions $\tilde{v}(t)$, $\tilde{x}(t)$, $\psi(t)$, $t_0 \leq t \leq \tau$, satisfy the maximum condition, the transversality conditions at the left and right endpoints, and the additional condition $H \equiv 0$, i.e. that $(\tilde{v}(t), \tilde{x}(t))$, $t_0 \leq t \leq \tau$, is an exceptional extremal process.

The meaning of this theorem is as follows. As a rule, for given M_0 and M_1 there do **not** exist exceptional extremal processes accomplishing the transfer from M_0 to M_1 (since the requirement $H \equiv 0$ constitutes excessive additional information, usually incompatible with extremality). If, however, exceptional processes do exist, then they are easy to find and to choose the best among them. Therefore, the theorem formulated above practically completely resolves the question of optimality in the case of a stable set M_1 . Let us note that this theorem can be formulated also in the following way:

Theorem 3. *If (under the assumption of stability of the set M_1) there exists no exceptional extremal process accomplishing the transfer from M_0 to M_1 , then, for optimality of the process $(v(t), x(t))$, $t_0 \leq t \leq t_1$, accomplishing the transfer from M_0 to M_1 , it is necessary and sufficient that it be extremal.*

The proof of the necessity of this condition (even without the assumption of the nonexistence of exceptional processes) coincides with the first part of the proof of Theorem 1 ⁽¹⁾ (the condition of strong stability of the set M_1 and the nondegeneracy condition are not used in this part of the proof). The sufficiency of the condition follows directly from Theorem 2.

In conclusion, we formulate one more proposition, which gives a sufficient condition for optimality that is sometimes convenient.

Proposition 3. *Let (under the assumption of stability of the set M_1) $(v(t), x(t))$, $t_0 \leq t \leq t_1$, be an extremal process that effects a transition from the set M_0 to the set M_1 . In addition, suppose that for $t_0 < \theta < t_1$ the hyperplane Γ_θ , passing through the point $x(\theta)$ and orthogonal to the vector $\psi(\theta)$, has no points in common with the set M_1 . Then the process $(v(t), x(t))$ is optimal.*

The proof of Proposition 3 coincides with the second part of the proof of Theorem 1 (1) (for $\psi(\theta)(x^* - x(\theta)) > 0$ for any point $x^* \in M_1$).

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2. V. G. Boltyanskii, *Mathematical Methods of Optimal Control*, 2nd ed., "Nauka," 1969.

Note: Figure translations are in progress. See original paper for figures.

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