ON SECOND SOLUTIONS OF BOUNDARY-VALUE PROBLEMS

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ON SECOND SOLUTIONS OF BOUNDARY-VALUE PROBLEMS

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In this paper, by methods of the theory of cones, boundary-value problems for nonlinear elliptic equations of second order are studied, as well as some boundary-value problems for ordinary differential equations of arbitrary order.

1. Let K be a cone in a Banach space E (^{1,2}), and let K^* be the set of positive linear functionals. Let a nonlinear operator F, acting and continuous from the Banach space E_1 into E ($E_1 \subset E$), and a linear operator A, acting and completely continuous from E into E_1, be positive: $FE_1 \subset K$, $AK \subset K$.

Suppose that E_1 is embedded in the Banach space E_0. We shall call the linear operator A E_0-contracting if there exist a linear functional $l \in K^*$, a vector $u_0 \in K$, and positive numbers λ and k such that $l(u_0) = 1$ and, for every $v \in K$, the inequalities

$$l(Av) \geq \lambda l(v), \qquad l(Av) \geq k\|Av\|_{E_0}. \tag{1}$$

Theorem 1. *Let the following conditions be satisfied:*

1) *A is an E_0-contracting operator;*
2) *$F\theta = \theta$, where θ is an isolated fixed point, completely continuous in E_1, of the vector field $\Phi u = u - AFu$, whose index is different from zero;*
3) *$l(Fu) \geq al(u) - b$ for every $u \in K$ (here a, b are positive constants, with $a\lambda > 1$);*
4) *for solutions belonging to $K \cap E_1$ of the family of operator equations*

$$u = (1-t)AFu + atAu \qquad (t \in [0,1])$$

the relation

$$\|u\|_{E_1} \leq \psi(\|u\|_{E_0}),$$

holds, where $\psi(s)$ is a continuous function.

Then the equation $u = AFu$ has at least one nonzero solution.

Theorem 1 is a development of the well-known fixed-point principle of M. A. Krasnosel' skii for an operator stretching a cone. The greatest difficulties arise in verifying the first condition. The choice of the space E_0 determines the character of the restrictions necessary for fulfilling the fourth condition, which is an abstract analogue of the classical L-condition of S. N. Bernstein.

2. We give examples of the use of Theorem 1 for proving the existence of a solution of some boundary-value problems.

Let Ω be a bounded domain in n-dimensional space with boundary Γ belonging to the class $A^{2,\alpha}$ (see ([3])) for some $\alpha \in (0,1)$.

Consider the differential equation

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial t_i} a_{ij}(t_1, \dots, t_n) \frac{\partial u}{\partial t_j} + a(t_1, \dots, t_n) u = f\left(t, u, \frac{\partial u}{\partial t_1}, \dots, \frac{\partial u}{\partial t_n}\right), \quad (2)$$

where $t = \{t_1, \dots, t_n\} \in \overline{\Omega}$, $\overline{\Omega} = \Omega + \Gamma$. In what follows it is assumed that the operator $\mathcal{L}$ is of elliptic type, that the coefficients $a_{ij}(t), a(t)$ are sufficiently smooth functions $(a_{ij}(t) \in C^{1+\alpha}, a(t) \in C^{\alpha})$, with $a(t) \geq 0$, and that the function $f(t, u, p_1, \dots, p_n)$ is defined and continuous for $t \in \overline{\Omega}$ and arbitrary $u, p_1, \dots, p_n$.

Let μ_1 denote the first eigenvalue of the operator $\mathcal{L}$ under the boundary condition

$$u_\Gamma = 0, \quad (3)$$

and let α_n, β_n be numbers satisfying the inequalities

$$\alpha_n (n-1) < n+1, \qquad \beta_n n < n+1, \qquad \alpha_n > 0, \qquad \beta_n > 0.$$

Theorem 2. Let the function $f(t, u, p)$ be nonnegative, have continuous partial derivatives with respect to each of its arguments, let

$$f(t, 0, 0, \dots, 0) = 0$$

and let the inequalities

$$f(t,u,p) \geq a_1 |u| - b_1 \qquad (b_1 > 0,\ a_1 > \mu_1),$$

$$f(t,u,p) \leq C(u)(1+p^2) \quad (n=1),$$

$$f(t,u,p) \leq C(1 + |u|^{\alpha_n} + |p|^{\beta_n}) \quad (n>1);$$

hold. Here $C(u)$ is a continuous function, C is a positive constant, and

$$|p| = \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}.$$

Then the boundary-value problem (2)–(3) has a nonzero solution belonging to the class $C^{2+\alpha}$.

An important role in the proof of Theorem 2 is played by

Lemma 1. Let $\varphi(t)$ be a nonnegative eigenfunction, and let $G(t,s)$ be the Green' s function of the operator $\mathcal{L}$ under the boundary condition (2). Then

$$\varphi(t) \geq k_1 \|G(t,s)\|_{L_p(\Omega)},$$

where k_1, p are positive constants, with $p(n-1) < n$.

The proof of Lemma 2 is based on estimates of the type of Giraud–Oleinik (3).

3. In this section we study the question of the existence of a solution of equation (2) satisfying, for $t \in \Gamma$, the boundary condition

$$\partial u/\partial \nu = \beta u. \tag{4}$$

Here $\partial u/\partial \nu$ is the derivative along the inward normal, and the function $\beta = \beta(t) \in C^{1+\alpha}$ is positive. Let μ_2 denote the first eigenvalue of the operator $\mathcal{L}$ under the boundary condition (4).

Theorem 3. Let the function $f(t_1, \dots, t_n, u, p_1, \dots, p_n)$ be nonnegative, have continuous partial derivatives with respect to each of its arguments, let

$$f(t, 0, \dots, 0) = 0$$

and satisfy the inequalities

$$f(t,u,p) \geq a_2|u| - b_2 \qquad (b_2 > 0,\ a_2 > \mu_2),$$

$$f(t,u,p) \leq C(\exp|u|^{\gamma_2} + |p|^{\delta_2}), \quad \text{for } n = 2,$$

$$f(t,u,p) \leq C(1 + |u|^{\gamma_n} + |p|^{\delta_n}), \quad \text{for } n > 2.$$

Here C, γ_n, δ_n are positive numbers, with

$$\gamma_2 < 1, \qquad \delta_2 < 2, \qquad \gamma_n(n-2) < n, \qquad \delta_n(n-1) < n \quad (n > 2).$$

Then the differential equation (2) has a positive solution satisfying the boundary condition (4) and belonging to $C^{2+\alpha}$.

4. Let us consider the problem of finding a solution of the ordinary differential equation

$$Mu \equiv \sum_{i=0}^{m} p_i(t)u^{(i)} = g(t, u, u', \dots, u^{(m-1)}), \tag{5}$$

satisfying the boundary condition

$$u(a_1) = u(a_2) = \dots = u(a_m) = 0. \tag{6}$$

Here $p_i(t)$ are functions defined and continuous on the interval $[0, T]$ ($|p_m(t)| > 0$), $g(t, u_1, \dots, u_m)$ is a function defined for $t \in [0, T]$, $-\infty < u_i < \infty$ $(i = 1, \dots, m)$; $a_1, a_2, \dots, a_m$ are certain numbers from the interval $[0, T]$, with $0 = a_1 < a_2 < \dots < a_m = T$. We shall assume that every solution of the equation $Mu = 0$ has on the interval $[0, T]$ no more than $n - 1$ zeros (counting multiplicities), and that the function $g(t, u_1, \dots, u_m)$ is nonnegative, continuous jointly in all variables together with the partial derivatives g_{u_i} $(i = 1, \dots, m)$.

Denote by $P(t, s)$ the Green' s function of the operator M under the boundary condition (6). It follows from the results of [4] that the integral equation

$$\mu \int_0^T |P(t, s)| z(t)\, dt = z(s)$$

has a nonnegative solution $y(s)$ for some $\mu = \mu_3 > 0$.

Lemma 2. There exists a positive constant k_2 such that, for all $s \in [0, T]$,

$$y(s) \geq k_2 \max_{t \in [0,T]} |P(t, s)|.$$

In the proof of Lemma 2, estimates of the Green' s function $P(t, s)$ established in [4] are essentially used.

Theorem 4. Suppose that $g(t, 0, \dots, 0) = 0$ and that, for any $t \in [0, T]$, $-\infty < u_i < \infty$ $(i = 1, \dots, m)$, the inequalities

$$c_3|u_1| - d_3 \leq g(t, u_1, \dots, u_m) \leq D(u_1) \left(1 + \sum_{i=1}^{m-1} |u_{i+1}|^{n/i - \varepsilon} \right), \tag{7}$$

hold, where $D(u_1)$ is a continuous function; c_3, d_3 are positive constants, with $c_3 > \mu_3$.

Then the boundary-value problem (5)–(6) has a nonzero solution.

5. We shall give conditions for the existence of a solution of the differential equation (5) satisfying the linear homogeneous boundary conditions

$$l_1(u) = l_2(u) = \dots = l_m(u) = 0 \tag{8}$$

We shall assume that under the boundary conditions (8) there exists a positive Green' s function $Q(t, s)$ of the operator M. Denote by μ_4 the first eigenvalue of the operator M under the boundary condition (8), and by $x(s)$ the corresponding first eigenfunction. The existence of an eigenfunction follows from the positivity of $Q(t, s)$.

Lemma 3. There exists a positive constant k_3 such that, for all $s \in [0, T]$,

$$x(s) \geq k_3 \left\{ \max_t \left| Q_{tn-2}^{(n-2)}(t, s) \right| + \sup_{\tau_1, \tau_2} \frac{\left| Q_{tn-2}^{(n-2)}(\tau_1, s) - Q_{tn-2}^{(n-2)}(\tau_2, s) \right|}{|\tau_1 - \tau_2|} \right\}.$$

An immediate consequence of Lemma 3 is

Theorem 5. *Let $g(t, 0, \dots, 0) = 0$ and let the inequality*

$$g(t, u_1, \dots, u_m) \geq c_4 |u_1| - \bar{d}_4 \quad (c_4 > \mu_4,\ \bar{d}_4 > 0)$$

hold.

Then the differential equation (5) has a positive solution satisfying the boundary conditions (8).

6. As was shown in (^{6}), the differential equation

$$\Delta u + |u|^r = 0$$

for $r \geq (n+2)/(n-2)$ $(n > 2)$ has no nonzero solutions satisfying the boundary condition (3). This example shows that the requirement of sublinear growth of the function $f(t, u, p)$ with respect to the variables $u, p_1, \dots, p_n$ is essential for the validity of the assertion of Theorem 2 and, in the general case, cannot be discarded.

Theorem 4 is an immediate generalization of the results of (^{5}). The scheme developed in the first section is applicable to systems of differential equations.

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