DEFINITION OF THE FIDUCIAL DISTRIBUTION FOR A MULTIVARIATE NORMAL POPULATION

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Abstract

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G. P. KLIMOV

DEFINITION OF THE FIDUCIAL DISTRIBUTION FOR A MULTIVARIATE NORMAL POPULATION

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Let $x_1, \dots, x_n$ be independent observations of a random variable (r.v.) from an r-dimensional normal population $N(\mu, A)$.

Case 1. A is known, μ is unknown. A sufficient statistic is

$$\bar{x} = \frac{1}{n}(x_1 + \dots + x_n).$$

For every $\bar{x}$, define the fiducial distribution (fid. d.) $P_{\bar{x}}$ of the parameter μ by the equality

$$P_\mu\{\mu \in S(\bar{x})\} = P_{\bar{x}}\{\mu \in S(\bar{x})\}$$

for every system of confidence (measurable) sets $S(\bar{x})$ satisfying the condition

$$\mu \in S(\bar{x}) \Longleftrightarrow \mu + \mu_0 \in S(\bar{x} + \mu_0)$$

for any r-dimensional vector μ_0.

Case 2. A is unknown, μ is known and $\mu = 0$. A sufficient statistic is

$$T = \sum_1^n x_k x_k'.$$

For almost every, with $n \geq r$, positive definite matrix T, define the fid. d. P_T of the parameter A by the equality

$$P_A\{A \in S(T)\} = P_T\{A \in S(T)\};$$

for any system of confidence (measurable) sets $S(T)$ satisfying the condition

$$A \in S(T) \Longleftrightarrow A_C \in S(T_C),$$

where $A_C = CAC'$, $T_C = CTC'$, and C is an arbitrary nonsingular matrix of dimension $r \times r$. Since any nonsingular real matrix C can be represented as the product of a finite number of matrices, each of which is either diagonal with positive elements on the main diagonal or orthogonal, the last condition means that the confidence sets are invariant with respect to the choice of the units of measurement and of the orthogonal coordinate system in which the sample variable is measured.

Case 3. A is unknown, μ is unknown. A sufficient statistic is the pair $(T, \bar{x})$, where

$$T = \sum_1^n (x_k - \bar{x})(x_k - \bar{x})'.$$

For every pair $(T, \bar{x})$, define the fid. d. $P_{(T,\bar{x})}$ of the parameter (A, μ) by the equality

$$P_{(A,\mu)}\{(A, \mu) \in S(T, \bar{x})\} = P_{(T,\bar{x})}\{(A, \mu) \in S(T, \bar{x})\}$$

for any system of confidence (measurable) sets $S(T, \bar{x})$ satisfying the condition:

$$(A, \mu) \in S(T, \bar{x}) \Longleftrightarrow (A_C, C\mu + \mu_0) \in S(T_C, C\bar{x} + \mu_0)$$

for any nonsingular matrix C and any r-dimensional vector μ_0. The last condition means that the confidence sets are invariant with respect to the choice of the coordinate system in which the sample variable is measured.

Now one may introduce the fiducial distribution of a sample variable as the distribution of the sample variable when the unobserved unknown parameter has the fiducial distribution corresponding to some value of a sufficient statistic.

Denote by $W^*(r, n, B)$ the distribution concentrated on the set A_r of positive definite matrices of dimension $r \times r$, with density

$$p(A) = \gamma_0(r, n) \frac{|B|^{n/2}}{|A|^{(n+r+1)/2}} \exp\left\{-\frac{n}{2}\operatorname{tr}(A^{-1}B)\right\}, \qquad n \geqslant r,$$

where $\gamma_0(r, n) = n^{nr/2}\gamma(r, n)$, and $\gamma(r, n)$ is the normalizing factor in the Wishart distribution. For every matrix $A \in A_r$, let $K(r, n, A)$ denote the distribution concentrated on the Euclidean space E_r, with density

$$p(x) = \gamma_1(r,n)\frac{1}{|A|^{1/2}}\left[1+\frac{(A^{-1}x,x)}{n}\right]^{-(n+1)/2}; \qquad \gamma_1(r,n) = (\pi n)^{-r/2}\times$$

$$\times\frac{\Gamma((n+1)/2)}{\Gamma((n-r+1)/2)}.$$

We shall mark the fiducial unobserved parameter and the fiducial sample variable with an asterisk as a superscript. Then, in case 1,

$$\mu^* \in N\left(\bar{x}, \frac{1}{n}A\right), \qquad x^* \in N\left(\bar{x}, \frac{n+1}{n}A\right)$$

(the symbol $\in$ means that, for example, μ^* has distribution $N\left(\bar{x}, \frac{1}{n}A\right)$).

In case 2,

$$A^* \in W^*(r,n,\hat{A}); \qquad x^*-\mu \in K(r,n,\hat{A}); \qquad \hat{A} = \frac{1}{n}\sum_1^n (x_k-\mu)(x_k-\mu)'.$$

In case 3,

$$\sqrt{n}(\mu^*-\bar{x}) \in K(r,n-1,S); \qquad A^* \in W^*(r,n-1,S);$$

$$x^*-\bar{x} \in K\left(r,n-1,\frac{n+1}{n}S\right); \qquad S = \frac{1}{n-1}\sum_1^n (x_k-\bar{x})(x_k-\bar{x})'.$$

Moscow State University
named after M. V. Lomonosov

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CITED LITERATURE

1. G. P. Klimov, DAN, 191, No. 4 (1970).

Note: Figure translations are in progress. See original paper for figures.