HOMOLOGICAL PROPERTIES OF MINIMAL COMPACTA IN THE MULTIDIMENSIONAL PLATEAU PROBLEM

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Abstract

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HOMOLOGICAL PROPERTIES OF MINIMAL COMPACTA IN THE MULTIDIMENSIONAL PLATEAU PROBLEM

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1. In modern works devoted to the study of the multidimensional Plateau problem, the main attention has so far been paid to questions connected with theorems on the existence of minimal compacta in one or another admissible class of surfaces. Such an approach is characteristic, for example, of works ($^{1-5}$) and others.

In (^{6}) the Plateau problem was posed and solved in a new class of compacta $O^k(A, \mathfrak{L}, \mathfrak{L}')$. The existence theorem for a minimal compactum in the class $O^k(A, \mathfrak{L}, \mathfrak{L}')$, formulated in that work, being a generalization of the corresponding results of Reifenberg (^{1,2}) and Morrey (^{3}), already makes it possible to raise the question of the global homological structure of minimal compacta and to distinguish at least three possible directions leading to the clarification of this question.

To the first direction one should assign the problem of describing the homology groups of minimal compacta; to the second, problems connected with the choice and study of a "minimal basis" in the groups $H_k(\mathfrak{M}^n; \mathfrak{G})$; to the third, the investigation of connections between the homological and topological properties of minimal compacta and the global structure of the set of their special points. The present note is devoted to the exposition of results obtained in the first and second directions. Some questions connected with the third direction will be discussed in a subsequent note.

2. By $\mathfrak{G}_C$ we shall denote the category of compact Abelian groups; by $\mathfrak{G}_F$, the category of linear spaces over a certain field F. As in (^{6}), we shall use the Čech-Aleksandrov homology theory.

Since, as it turns out, the topological properties of minimal compacta in the classes $O^k(A, \mathfrak{L}, \mathfrak{L}')$ (see the definition in (^{6})) change sharply when the subgroups $\mathfrak{L}$ and $\mathfrak{L}'$ are changed, we shall distinguish and consider separately the following four cases, which together exhaust all possible classes $O^k(A, \mathfrak{L}, \mathfrak{L}')$:
a) $O^k(A, \mathfrak{L}, 0)$, $\mathfrak{L} \neq 0$;

b) $O^k(A, \mathfrak{L}, \mathfrak{L}')$, $\mathfrak{L} \neq 0$, $\mathfrak{L}' \neq 0$;
c) $O^k(A, 0, \mathfrak{L}')$, $\mathfrak{L}' \neq 0$;
d) $O^k(\emptyset, 0, \mathfrak{L}')$, $\mathfrak{L}' \neq 0$.

Minimal compacta in these classes (their existence and local properties were studied in (6)) will be denoted respectively by $X_0^m, Z_0^m, T_0^m, Y_0^m$. Let us consider the homomorphisms induced by embeddings of these compacta into the enveloping manifold $\mathfrak{M}^n$:

$$(i_Z)_* : H_k(Z_0^m; \mathfrak{G}) \to H_k(\mathfrak{M}^n; \mathfrak{G}); \quad (i_T)_* : H_k(T_0^m; \mathfrak{G}) \to H_k(\mathfrak{M}^n; \mathfrak{G}); \quad (i_Y)_* : H_k(Y_0^m; \mathfrak{G}) \to H_k(\mathfrak{M}^n; \mathfrak{G}).$$

Here the number $k \geq 3$ denotes the dimension of the minimal compacta.

By $\mathfrak{M}^n$ in what follows we denote a compact, closed, Riemannian manifold of class at least C^4.

3. Theorem 1. *Let* $A \subset \mathfrak{M}^n$ *be an arbitrary but fixed compactum;* $\mathfrak{G} \in \mathfrak{G}_F$, $\dim \mathfrak{G} = 1$. *Suppose that* $H_k(A; \mathfrak{G}) = 0$. *Then:*

I. $H_k(X_0^m; \mathfrak{G}) = 0$ *(for any subgroup* $\mathfrak{L} \neq 0$*).*

II. The homomorphisms $(i_Z)_*, (i_T)_*, (i_Y)_*$ *are monomorphisms.*

Remark. From Assertion I it follows, in particular, that the minimal "film" X_0^m cannot realize a nontrivial cycle in the group $H_k(\mathfrak{M}^n; \mathfrak{G})$ (for the definition of realization see (6)).

Both assumptions of Theorem 1 are essential. Dropping the condition $H_k(A; \mathfrak{G}) = 0$ immediately leads to elementary counterexamples to both Assertions I and II. For example, if the condition $\dim \mathfrak{G} = 1$ is dropped, it is easy to construct a minimal compactum Y_0^m, even a smooth submanifold in $\mathfrak{M}^n$, such that $\operatorname{Ker}(i_Y)_* \neq 0$ (item II). If, for example, one sets $\mathfrak{G} = U = \mathbb{R}^1 (\operatorname{mod} 1)$, then one can construct a compactum X_0^m having a nontrivial group $H_k(X_0^m; U)$ and, moreover, realizing a cycle in the group $H_k(\mathfrak{M}^n; U)$.

For this purpose, consider the manifold $\mathfrak{M}^{2s+1} = S^1 \times \mathbb{R}^{2s}$ and take in $\mathfrak{M}^{2s+1}$ a compact pair (X_0, A), where $X_0 = \mathbb{R}^{2s}$, $A = \mathbb{R}P^{2s-1}$, and $A \subset X_0$ as the $(2s-1)$-dimensional skeleton in the CW-complex $\mathbb{R}P^{2s}$. Then $H_{2s-1}(X_0; U) = 0$; $H_{2s-1}(A; U) = U$, whence it follows that $X_0 \in O^k(A, U, 0)$ (here $k = 2s$ and $U = \mathfrak{L} = H_{k-1}(A; U)$). It is easy to show that the compactum X_0 is a minimal compactum X_0^m in the class $O^k(A, U, 0)$. On the other hand, it is obvious that $H_k(X_0^m; U) = \mathbb{Z}_2$ and, moreover, the compactum $X_0^m \subset \mathfrak{M}^{2s+1}$ realizes the subgroup $\mathbb{Z}_2 \subset H_k(\mathfrak{M}^n; U)$ (item I).

4. **Definition 1.** We shall call a **minimal carrier** $\operatorname{Car}^m(A, \mathfrak{L}, \mathfrak{L}')$ of the triple $(A, \mathfrak{L}, \mathfrak{L}')$ (where $A \subset \mathfrak{M}^n$, $\mathfrak{L} \subset H_{k-1}(A)$, $\mathfrak{L}' \subset H_k(\mathfrak{M}^n)$) a minimal compactum $X_0 \in O^k(A, \mathfrak{L}, \mathfrak{L}')$. If $A = \emptyset$, then $\operatorname{Car}^m(\emptyset, 0, \mathfrak{L}') = \operatorname{Car}^m(\mathfrak{L}') = Y_0^m$; if $\mathfrak{L}' = 0$, then $\operatorname{Car}^m(A, \mathfrak{L}, 0) = \operatorname{Car}^m(\mathfrak{L}) = X_0^m$.

Definition 2. We shall call the **k-dimensional Hausdorff measure** $\Lambda^k(\mathfrak{L}')$ of the subgroup $\mathfrak{L}' \subset H_k(\mathfrak{M}^n; \mathfrak{G})$ the number $\Lambda^k(\mathrm{Car}^m(\mathfrak{L}')) = d(\emptyset, 0, \mathfrak{L}')$.

Definition 3. We shall call the **k-dimensional Hausdorff measure** $\Lambda^k(\mathfrak{L}')$ of the subgroup $\mathfrak{L}' \subset H_k(\mathfrak{M}^n; \mathfrak{G})$ the number $\Lambda^k(\mathrm{Car}^m(\mathfrak{L}')) = d(A, \mathfrak{L}, 0)$.

Theorem 2. Let $A \subset \mathfrak{M}^n$, $\mathfrak{G} \in \mathfrak{G}_F$, $\dim \mathfrak{G} = 1$, $H_k(A; \mathfrak{G}) = 0$. Consider a nonempty class $O^k(A, \mathfrak{L}, \mathfrak{L}')$, where $\mathfrak{L}' \neq 0$. Then in the subgroup $\mathfrak{L}'$ there exists a subgroup $\widetilde{\mathfrak{L}}'$ such that:

I. Codim $\widetilde{\mathfrak{L}}' = 1$ in $\mathfrak{L}'$.
II. $d(A, \mathfrak{L}, \mathfrak{L}') > d(A, \mathfrak{L}, \widetilde{\mathfrak{L}}')$.

Corollary 1. Suppose that all assumptions of Theorem 2 are satisfied. Then any subgroup $\mathfrak{L}' \subset H_k(\mathfrak{M}^n; \mathfrak{G})$ contains a decreasing chain of subgroups

$$\mathfrak{L}' = \mathfrak{L}'_0 \supset \mathfrak{L}'_1 \supset \mathfrak{L}'_2 \supset \cdots \supset \mathfrak{L}'_N,$$

where:

I. Codim $\mathfrak{L}'_{p+1} = 1$ in $\mathfrak{L}'_p$ for $0 \leq p \leq N-1$, $\dim \mathfrak{L}'_N = 1$.

II. $\Lambda^k(\mathfrak{L}'_p) > \Lambda^k(\mathfrak{L}'_{p+1})$, $0 \leq p \leq N-1$.

The assertion of Corollary 1 corresponds to the intuitive notion that, as the number of cycles in a subgroup decreases, the Hausdorff measure of its minimal carrier should, generally speaking, strictly decrease. Theorem 2, like all subsequent assertions of the present note, is proved with the aid of Theorem 1.

5. Corollary 1 admits a dual reformulation in terms of the class $O^k(A, \mathfrak{L}, 0)$.

Corollary 2. Let $A \subset \mathfrak{M}^n$, $\mathfrak{G} \in \mathfrak{G}_F$, $\dim \mathfrak{G} = 1$, $H_k(A; \mathfrak{G}) = 0$. Consider a nonempty class $O^k(A, \mathfrak{L}, 0)$ and an arbitrary subgroup $\mathfrak{L} \subset H_{k-1}(A; \mathfrak{G})$ such that $\Lambda^k(\mathfrak{L}) < \infty$. Then there exists a decreasing chain of subgroups

$$\mathfrak{L} = \mathfrak{L}^0 \supset \mathfrak{L}^1 \supset \mathfrak{L}^2 \supset \cdots$$

such that:

I. Codim $\mathfrak{L}^{p+1} = 1$ in $\mathfrak{L}^p$.
II. $\Lambda^k(\mathfrak{L}^p) > \Lambda^k(\mathfrak{L}^{p+1})$ for $p \geq 0$.

Remark. Simple examples show that, in the terms of the class $O^k(A, \mathfrak{L}, \mathfrak{L}')$, with variable subgroup $\mathfrak{L}$ and $\mathfrak{L}' \neq 0$, an assertion dual to Theorem 2 generally cannot be formulated; this once again emphasizes the special nature of the class $O^k(A, \mathfrak{L}, 0)$ in comparison with the three "realizing classes," for which $\mathfrak{L}' \neq 0$.

6. **Definition 4.** We shall say that a subgroup $\mathfrak{L}' \subset H_k(\mathfrak{M}^n; \mathfrak{G})$ admits an **exact minimal realization** if

$$\mathfrak{L}' \cong H_k(\mathrm{Car}^m(\mathfrak{L}'); \mathfrak{G}).$$

Let us note that, by Theorem 1, there is always an isomorphism

$$H_k(Y_0^m; \mathfrak{G}) \cong \mathrm{Im}(i_Y)_*.$$

From the geometric point of view, Definition 4 means that the compact set $\mathrm{Car}^m(\mathfrak{L}')$ carries no "superfluous" cycles. Examples show that the general case is the situation in which a subgroup $\mathfrak{L}' \subset H_k(\mathfrak{M}^n; \mathfrak{G})$ does not admit an exact minimal realization. Thus, for example, if one considers the group $H_1(T^2; \mathbf{Q})$, where $\mathbf{Q}$ is the field of rational numbers and T^2 is the 2-dimensional torus with the Euclidean metric (as distinct from the metric induced by the embedding $T^2 \subset \mathbf{R}^3$), then an exact minimal realization is admitted only by four one-dimensional subgroups of the form: $\{a\}$, $\{b\}$, $\{a+b\}$, $\{a-b\}$, where a and b are the generators corresponding to trajectories on the torus of types $(0,1)$ and $(1,0)$.

Nevertheless, despite their "scarcity," the subgroups $\mathfrak{L}'$ admitting an exact realization have a number of important properties.

7. **Theorem 3.** Let $A \subset \mathfrak{M}^n$, $\mathfrak{G} = Z_p$, $p \neq 0$, be simple; $H_k(A; Z_p) = 0$. Consider a nonempty class $O^k(A, \mathfrak{L}, \mathfrak{L}')$, $\mathfrak{L}' \neq 0$, and $d(A, \mathfrak{L}, \mathfrak{L}') < \infty$. Let $r = \dim \mathfrak{L}'$. Then, if $r > 1$, in the subgroup $\mathfrak{L}'$ one can choose a basis $e_1, e_2, \dots, e_r$ such that:

I.

$$d(A, \mathfrak{L}, \{e_j\}) < d(A, \mathfrak{L}, \mathfrak{L}') \quad \text{for } 1 \leq j \leq r;$$

$$d(A, \mathfrak{L}, \{e_j\}) < d(A, \mathfrak{L}, \{e_1, e_2, \dots, e_j\}) \leq d(A, \mathfrak{L}, \mathfrak{L}') \quad \text{for } 1 < j \leq r.$$

II.

$$(\mathrm{Im}\, H_k(X_{0j})) \cap \mathfrak{L}' = \{e_j\},$$

where

$$X_{0j} = \mathrm{Car}^m(A, \mathfrak{L}, \{e_j\}),$$

and the homomorphism

$$H^k(X_{0j}; Z_p) \to H_k(\mathfrak{M}^n; Z_p)$$

is induced by the embedding.

III. If an element $a \in \mathfrak{L}'$ has the form

$$\sum_{p=1}^{s} a_{i_p} e_{i_p},$$

where all $a_{i_p} \neq 0$, then

$$d(A, \mathfrak{L}, \{a\}) \geq \max_{1 \leq p \leq s} d(A, \mathfrak{L}, \{e_{i_p}\}).$$

Here $\{a, b, \dots, c\}$ denotes the subgroup in $H_k(\mathfrak{M}^n; \mathfrak{G})$ generated by the elements $a, b, \dots, c$.

Corollary 3. Let $\mathfrak{M}^n$ be a compact, closed Riemannian manifold, $\mathfrak{M}^n \in C^4$, $\mathfrak{G} = Z_p$, $p \neq 0$, be simple, $r \neq \dim H_k(\mathfrak{M}^n; Z_p)$. Then, if $r > 1$, in the group $H_k(\mathfrak{M}^n; Z_p)$ there exists a basis $e_1, e_2, \dots, e_r$ such that:

I.

$$\Lambda^k(\{e_j\}) < \Lambda^k(H_k(\mathfrak{M}^n)), \quad 1 \leq j \leq r;$$

$$\Lambda^k(\{e_j\}) < \Lambda^k(\{e_1, e_2, \dots, e_j\}) \leq \Lambda^k(H_k(\mathfrak{M}^n)), \quad 1 < j \leq r.$$

II. All one-dimensional subgroups $\{e_j\}$, $1 \leq j \leq r$, admit an exact minimal realization.

III. If an element $a \in H_k(\mathfrak{M}^n; Z_p)$ has the form:

$$\sum_{p=1}^{s} a_{i_p} e_{i_p},$$

where all

$$a_{i_p} \neq 0,$$

then

$$\Lambda^k(\{a\}) \geq \max_{1 \leq p \leq s} \Lambda^k(\{e_{i_p}\}).$$

Remark. Assertion III is not a consequence of item II: it is easy to construct an example in which

$$\Lambda^k(\{a + b\}) < \min[\Lambda^k(\{a\}), \Lambda^k(\{b\})],$$

although the one-dimensional subgroups $\{a\}$ and $\{b\}$ admit exact minimal realizations.

8. We now consider the class $O^k(A, \mathfrak{L}, 0)$, $\mathfrak{L} \neq 0$. It turns out that there exists a theorem dual to Corollary 3.

Theorem 4. Let $\mathfrak{M}^n$ be a compact, closed Riemannian manifold, $\mathfrak{M}^n \in C^4$, $A \subset \mathfrak{M}^n$, $\mathfrak{G} = Z_p$, $p \neq 0$, be simple; $H_k(A; Z_p) = 0$. Suppose

$$O^k(A, \mathfrak{L}, 0) \neq \emptyset, \qquad d(A, \mathfrak{L}, 0) < \infty, \qquad \mathfrak{L} \subset H_{k-1}(A; Z_p),$$

$$r = \dim \mathfrak{L}, \qquad 1 < r < \infty.$$

Then in the subgroup $\mathfrak{L}$ one can choose a basis $e_1, e_2, \dots, e_r$ such that:

I.

$$\Lambda^k(\{e_j\}) < \Lambda^k(\mathfrak{L}), \quad 1 \leq j \leq r;$$

$$\Lambda^k(\{e_j\}) < \Lambda^k(\{e_1, e_2, \dots, e_j\}) \leq \Lambda^k(\mathfrak{L}), \quad 1 < j \leq r.$$

II.

$$(\operatorname{Ker} j_{p*}) \cap \mathfrak{L} = \{e_p\},$$

where $j_p : A \to \operatorname{Car}^m(\{e_p\})$ is the embedding.

III. If the element $\alpha \in \mathfrak{L}$ has the form

$$\sum_{p=1}^{s} a_{i_p} e_{i_p},$$

where all $a^0_{i_p} \neq 0$, then

$$\Lambda^k(\{\alpha\}) \geq \max_{1 \leq p \leq s} \Lambda^k(\{e_{i_p}\}).$$

Item II in Theorem 4 means that the algebraic boundary of the compact pair $(\operatorname{Car}^m(A, \{e_p\}, 0); A)$ (see definition in (6)) intersects the subgroup $\mathfrak{L}$ only in the one-dimensional subgroup $\{e_p\}$. If we now put $\mathfrak{L} = \bar{H}_{k-1}(A; Z_p)$, then we obtain a theorem on a basis already in the whole group $\bar{H}_{k-1}(A, Z_p)$, and each minimal compactum $\operatorname{Car}^m(\{e_p\})$ spans exactly one one-dimensional subgroup $\{e_p\} \subset \bar{H}_{k-1}(A; Z_p)$.

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