ON THE CHEBYSHEV POINT OF A SYSTEM OF PLANES IN A BANACH SPACE

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Abstract

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ON THE CHEBYSHEV POINT OF A SYSTEM OF PLANES IN A BANACH SPACE

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We consider the problem of the minimum of a convex, continuous, and nonnegative functional $f(x)$, defined on a Banach space X, for which the set

$$L = \{x \mid f(x) = f(0),\ x \in X\}$$

is linear, and moreover

$$f(z + x) = f(z)$$

for every element $x \in L$. We shall call the set L the subspace of constancy of the functional $f(x)$. An example of a functional of this type is

$$\sup_{i \in I} \rho(x, H_i),$$

where $H_i = x_i + L_i,\ i \in I$, L_i is a subspace of X, $x_i \in X$, and $\rho(x, H_i)$ is the distance from x to H_i. For this functional the subspace of constancy is

$$\bigcap_{i \in I} L_i,$$

and a point $x^* \in X$ for which

$$\sup_{i \in I} \rho(x^*, H_i) = \inf_{x \in X} \sup_{i \in I} \rho(x, H_i)$$

is called a Chebyshev point for the system of planes $H_i,\ i \in I$. Algorithms for finding such a point in the case of a finite-dimensional space were developed by S. I. Zukhovitskii ([1]); a proof of its existence for finite $I = \{1, \dots, n\}$, an arbitrary Banach space, and $L_i,\ i \in I$, of unit defect (index), was given by A. L. Garkavi ([2]).

Another important example of a functional possessing a subspace of constancy is $\|Ux - y\|$, where U is a linear and continuous operator from X into a Banach space Y, and y is a fixed element of Y; for this functional the subspace of constancy is the set of zeros of the operator U.

We state the following condition:

(1). For every number $r > 0$ there exists a number $R > 0$ such that from the inequality $f(x) \leq r$ it follows that $\rho(x, L) \leq R$.

For the functional

$$\sup_{i\in I} \rho(x, H_i)$$

this condition is formulated as follows:

(1+). For every number $r > 0$ there exists a number $R > 0$ such that from the inequalities

$$\rho(x, L_i) \leq r, \quad i \in I,$$

it follows that

$$\rho\left(x, \bigcap_{i\in I} L_i\right) \leq R.$$

For condition (1) to hold for the functional $\|Ux - y\|$, it is sufficient, and if the subspace of U is quotient-reflexive,* also necessary, that the range of the operator U be closed.

Theorem 1. In order that condition (1+) hold for the subspaces $L_1, \dots, L_m$, it is sufficient (and for $m = 2$ also necessary) that the sum of the subspaces

$$\sum_{i=1}^{m} L_i$$

be closed.

For $m > 2$ condition (1+) is no longer sufficient for the closedness of the sum of the subspaces.

Theorem 2. If for a convex, continuous, and nonnegative functional $f(x)$ the subspace of constancy is quotient-reflexive and

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* A subspace $L \subset X$ is called quotient-reflexive ([3]) if the quotient space X/L is reflexive.

satisfies condition (1), then there exists a point $x^* \in X$ such that

$$\inf_{x\in X} f(x) = f(x^*).$$

From Theorems 1 and 2 there follows

Corollary. Let $\bigcap_{i=1}^{m} L_i$ be a factor-reflexive subspace. In order that, for any planes $H_i = x_i + L_i,\ x_i \in X,\ i = 1, \dots, m$, there exist a Chebyshev point, it is sufficient (and for $m = 2$ also necessary) that the sum $\sum_{i=1}^{m} L_i$ be closed (that condition (1+) be satisfied).

For $m > 2$ these conditions are not necessary.

Let, further, A_i be a linear and continuous operator from X into the Banach space Y_i; D_i, N_i, R_i respectively the domain of definition, the null set, and the range of the operator $A_i,\ i = 1, \dots, m,\ \bigcap_{i=1}^{m} D_i \neq \Lambda$.

Denote by $\varphi(\ldots)$ some norm in m-dimensional coordinate space.

Theorem 3. If the space $\bigcap_{i=1}^{m} N_i$ is factor-reflexive and the sets $\sum_{i=1}^{m} N_i$, $\bigcap_{i=1}^{m} D_i$, R_i, $i = 1, \ldots, m$, are closed, then in $\bigcap_{i=1}^{m} D_i$ there exists a point delivering the minimum to the functional

$$\varphi(\|A_1 x - y_1\|, \ldots, \|A_{mx} - y_m\|),$$

whatever fixed elements $y_i \in Y_i$, $i = 1, \ldots, m$, may be.

Theorem 4. Whatever the system of functionals $f_1, \ldots, f_n \subset X^*$ and the system of numbers $c_1, \ldots, c_n, d$, where $d \geq 0$, there exists a system of points $x_1^*, \ldots, x_m^*$ (m fixed), for which

$$\max_{1\leq i\leq n} \min_{1\leq j\leq m} |f_i(x_j^*) - c_2| = \inf_{\substack{x_1,\ldots,x_m \subset X \\ \max_{i,j\leq m} \|x_i - x_j\| \leq d}} \max_{1\leq i\leq n} \min_{1\leq j\leq m} |f_i(x_j) - c_i|.$$

Hence, for $m = 1$ and $d = 0$ (as, however, also from the corollary and Theorem 3), there follows the existence of a Chebyshev point of a finite system of hyperplanes. The problem of finding such a point in the present case can be reduced to solving a certain system of equations. Namely, let the system of functionals $f_1, \ldots, f_n \subset X^*$ be linearly independent,

$$f_{n+i} = \sum_{k=1}^{n} \lambda_k^{(i)} f_k,$$

$i = 1, \ldots, m$, and let the system of numbers $c_1, \ldots, c_{n+m}$ be arbitrary. Denote

$$\mu_i = c_{n+i} - \sum_{k=1}^{n} \lambda_k^{(i)} c_k, \qquad i = 1, \ldots, m,$$

and suppose, for example, that $\mu_1 \neq 0$. By the method of linear programming we find the quantity

$$N = \max\left(\sum_{k=1}^{n} \lambda_k^{(1)} x_k - x_{n+1}\right),$$

where the maximum is taken over all systems of numbers $x_1, \ldots, x_{n+m}$ satisfying the equation

$$\sum_{k=1}^{n} \left(\lambda_k^{(i)} - \frac{\mu_i}{\mu_1}\lambda_k^{(1)}\right) x_k + \frac{\mu_i}{\mu_1} x_{n+1} - x_{n+i} = 0, \qquad i = 2, \ldots, m,$$

and the inequalities $|x_k| \leq 1$, $k = 1, \ldots, n+m$; moreover, we find a system of numbers $x_1^0, \ldots, x_{n+m}^0$, at which the quantity N is attained.

Theorem 5. Every solution of the determined system

$$f_\nu(x) = c_\nu - \left(\sum_{k=1}^{n} \lambda_k^{(1)} c_k - c_{n+1}\right) \frac{x_\nu^0}{N}, \qquad \nu = 1, \dots, n,$$

is a Chebyshev point for the system of hyperplanes $f_j(x) = c_j$, $j = 1, \dots, n,\ n+1, \dots, n+m$. If the system of numbers $x_1^0, \dots, x_{n+m}^0$ is unique, then the converse is also true.

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REFERENCES

1. S. I. Zukhovitskii, *DAN*, **139**, No. 3 (1961).

2. A. L. Garkavi, in: *Collected Mathematical Papers*, V. V. Kuibyshev Military Engineering Academy, 1968.

3. A. L. Garkavi, *Mat. sbornik*, **62** (104), No. 1 (1963).

Note: Figure translations are in progress. See original paper for figures.