ON THE DISCRETENESS OF THE SPECTRUM OF NON-SELF-ADJOINT OPERATORS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.43

MATHEMATICS

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ON THE DISCRETENESS OF THE SPECTRUM OF NON-SELF-ADJOINT OPERATORS

(Presented by Academician L. S. Pontryagin, 25 IX 1969)

In the present note we give theorems that make it possible to reduce the question of the discreteness of the spectrum of certain non-self-adjoint operators to the question of the discreteness of the spectrum of self-adjoint operators.

1. Let D_0 be a linear manifold dense in a Hilbert space H; let L_1 and L_2 be operators symmetric on D_0. Suppose that the operator $L_0' = L_1 + iL_2$ has a closure L_0. Then the following theorems are valid.

Theorem 1. *If at least one of the operators $L_1, L_2, L_1 + L_2, L_1 - L_2$ is semibounded and its semibounded self-adjoint extension has a discrete spectrum, then the kernel of the spectrum of the operator L_0 is discrete.*

Proof. Let, for example, $(L_1 y, y) \geq (y, y)$ for $y \in D_0$, and let $\widetilde{L}_1$ be a semibounded self-adjoint extension of the operator L_1. The set

$$\widetilde{E}_1 = [y : (\widetilde{L}_1 y, y) \leq 1,\ y \in D(\widetilde{L}_1)]$$

is compact in H, since $\widetilde{L}_1$ has a discrete spectrum. All the more compact is the set

$$E_1 = [y : (L_1 y, y) \leq 1,\ y \in D_0].$$

The set

$$E_0 = [y : |(L_0 y, y)| \leq 1,\ y \in D_0] \subset E_1,$$

and therefore it is also compact. Suppose that L_0 has a point λ_0 of the continuous spectrum; then, for any $\varepsilon > 0$, there exists a noncompact sequence $\{\varphi_i\}$, $\varphi_i \in D_0$, such that $1/2 \leq \|\varphi_i\| \leq 2$ and

$$\|L_0 \varphi_i - \lambda_0 \varphi_i\| \leq \varepsilon. \tag{1}$$

Put $\varepsilon = 1$. From the last inequality we have

$$\|L_0 \varphi_i\| \leq |\lambda_0| \|\varphi_i\| + 1 \leq 2|\lambda_0| + 1. \tag{2}$$

Obviously, the sequence $\{\varphi_i'\}$, where $\varphi_i' = \varphi_i/2(2|\lambda_0|+1)$, is noncompact together with the sequence $\{\varphi_i\}$, but, by virtue of inequality (2), it belongs to E_0. We have obtained a contradiction to the compactness of the set E_0. The remaining cases are easily reduced to the one proved.

The theorem just proved generalizes Theorem 4 of Ch. II of the paper (^{6}).

Theorem 2. *If*

$$(L_1 y, y) \geq (y, y) \quad \text{for } y \in D_0; \qquad (L_2 y, y) \geq (y, y) \quad \text{for } y \in D_0, \tag{3}$$

and the operator L_0 has no residual spectrum, then the operator L_0 has a discrete spectrum if and only if the set

$$E_0 = [y : |(L_0 y, y)| \leq 1,\ y \in D_0]$$

is compact in H.

Proof of sufficiency. Suppose the set E_0 is compact; then the set

$$E = [y : (L_0 y, y) \leq 1,\ y \in D(L_0)] \subset \widetilde{E}_0$$

is also compact, where $\widetilde{E}_0$ is the closure of the set E_0 in H. The set

$$E' = [y : \|L_0 y\| \leq 1,\ y \in D(L_0)] \subset E$$

is consequently also compact in H; therefore the operator L_0^{-1} is completely continuous, and hence the spectrum of the operator L_0 is discrete (the existence of the operator L_0^{-1} follows from the conditions (3) and from the absence of residual spectrum for the operator L_0).

Proof of necessity. Suppose that the spectrum of the operator L_0 is discrete; then the inverse operator $L_0^{-1} = A$ is completely continuous. From

from condition (3) we obtain that $A = A_1 - iA_2$, where A_1 and A_2 are positive self-adjoint completely continuous operators.

Suppose that the set E_0 is not compact; then there is a noncompact sequence $\{y_n\} \subset E_0$. From (3) it follows that $\|y_n\| \leq 1$. Denote $L_0 y_n = \varphi_n$; then

$$(L_0 y_n, y_n) = (\varphi_n, A\varphi_n) = (A_1 \varphi_n, \varphi_n) - i(A_2 \varphi_n, \varphi_n). \tag{4}$$

Since A_1 and A_2 are positive self-adjoint completely continuous operators, there exist orthonormal systems $\{e_i\}$ and $\{g_i\}$ such that

$$A_1 \varphi_n = \sum_1^\infty \lambda_i(A_1)(\varphi_n, e_i)e_i, \qquad A_2 \varphi_n = \sum_1^\infty \lambda_i(A_2)(\varphi_n, g_i)g_i, \tag{5}$$

where $\lambda_i(A_1) > 0$, $\lambda_i(A_2) > 0$ and tend to zero as $i \to \infty$. We shall prove that for every $\nu > 0$ there is an n such that at least one of the inequalities

$$\left\|\sum_{\nu}^{\infty}\lambda_i(A_1)(\varphi_n,e_i)e_i\right\|\geq\varepsilon_0,\qquad\left\|\sum_{\nu}^{\infty}\lambda_i(A_2)(\varphi_n,g_i)g_i\right\|\geq\varepsilon_0,\tag{6}$$

holds, where $\varepsilon_0>0$ is independent of both n and ν.

Suppose that the assertion is false; then for every $\varepsilon>0$ there is a ν such that

$$\left\|\sum_{\nu}^{\infty}\lambda_i(A_1)(\varphi_n,e_i)e_i\right\|<\varepsilon,\qquad\left\|\sum_{\nu}^{\infty}\lambda_i(A_2)(\varphi_n,g_i)g_i\right\|<\varepsilon\tag{7}$$

for all n. Then

$$\begin{aligned}A\varphi_n&=A_1\varphi_n-iA_2\varphi_n\\&=\left[\sum_{1}^{\nu}\lambda_i(A_1)(\varphi_n,e_i)e_i-i\sum_{1}^{\nu}\lambda_i(A_2)(\varphi_n,g_i)g_i\right]+\\&\quad+\left[\sum_{\nu}^{\infty}\lambda_i(A_1)(\varphi_n,e_i)e_i-i\sum_{\nu}^{\infty}\lambda_i(A_2)(\varphi_n,g_i)g_i\right]=g_n^{\nu}+\psi_n^{\nu},\end{aligned}\tag{8}$$

where

$$g_n^{\nu}=\sum_{1}^{\nu}\lambda_i(A_1)(\varphi_n,e_i)e_i-i\sum_{1}^{\nu}\lambda_i(A_2)(\varphi_n,g_i)g_i,$$

$$\psi_n^{\nu}=\sum_{\nu}^{\infty}\lambda_i(A_1)(\varphi_n,e_i)e_i-i\sum_{1}^{\nu}\lambda_i(A_2)(\varphi_n,g_i)g_i.$$

By inequalities (7) and equality (8), $\|\psi_n^{\nu}\|\leq 2\varepsilon$, and $\|g_n^{\nu}\|\leq 1+2\varepsilon$. The set $\{g_n^{\nu}\}$ belongs to a finite-dimensional space and is bounded; therefore there exists a finite ε-net $\{\eta_i\}$ for it, and since $\|\psi_n^{\nu}\|\leq 2\varepsilon$, the set $\{\eta_i\}$ forms a 3ε-net of the set $\{y_n\}=\{A\varphi_n\}$. But this contradicts the noncompactness of the sequence $\{y_n\}$. Suppose, for example, that the first of inequalities (6) is satisfied; then

$$1\geq(A_1\varphi_n,\varphi_n)=\sum_{1}^{\infty}\lambda_i(A_1)|(\varphi_n,e_i)|^2\geq\sum_{\nu}^{\infty}\lambda_i(A_1)|(\varphi_n,e_i)|^2\geq$$

$$\geq\frac{1}{\lambda_\nu(A_1)}\sum_{\nu}^{\infty}\lambda_i^2(A_1)|(\varphi_n,e_i)|^2\geq\frac{1}{\lambda_\nu(A_1)}\varepsilon_0.$$

The last inequality contradicts the fact that $\lambda_\nu(A_1)\to 0$ as $\nu\to\infty$. The theorem is proved.

Remark. Under the conditions of Theorem 2 the inequalities

$$\frac{1}{\sqrt{2}}((L_1+L_2)y,y) \le |(L_0y,y)| \le ((L_1+L_2)y,y), \tag{9}$$

hold when $y \in D_0$; therefore the set E_0 is compact if and only if the set $E_1 = [y:\ ((L_1+L_2)y,y) \le 1,\ y \in D_0]$ is compact. Hence it is easy to obtain that the following assertion is valid: if the conditions of Theorem 2 are satisfied and the closure of the operator L_1+L_2, the operator $\overline{L_1+L_2}$, has finite defect indices, then the spectrum of the operator L_0 is discrete if and only if the operator $\overline{L_1+L_2}$ has a self-adjoint extension with discrete spectrum.

2. Let I be some conjugation operator in the Hilbert space H. A linear operator A with domain of definition $D(A)$ dense in H will be called I-symmetric if, for any φ and ψ in $D(A)$, the equality $(A\varphi, I\psi) = (\varphi, IA\psi)$ holds, equivalent to the relation $A \subset IA^*I$. From the latter relation it follows that an I-symmetric operator admits a closure. If $A = IA^*I$, then the operator A is called I-self-adjoint (^{1}). In (^{5}) it is shown that an I-symmetric operator admits an I-self-adjoint extension.

Combining the methods of P. S. Ismagilov (^{3}) and Levinson (^{2}), it is easy to prove that the minimal operator L_0, generated by the differential expression

$$l[u] = -\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial y}\right) + gu \tag{10}$$

in $L_2(E_2)$, where a, b, g are complex-valued locally integrable functions of the variables x and y, $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$, is I-self-adjoint if

$$\iint_{E_2} \frac{dx\,dy}{\sqrt{|a|M}} = +\infty, \qquad \iint_{E_2} \frac{dx\,dy}{\sqrt{|a|M}} = +\infty; \tag{11}$$

here $M(r)$ $(r = \sqrt{x^2+y^2})$ is a positive nondecreasing function on $(0,\infty)$;

$$\frac{\sqrt{|a|}\,\partial M/\partial x}{\sqrt{M^3}} < +\infty, \qquad \frac{\sqrt{|b|}\,\partial M/\partial y}{\sqrt{M^3}} < +\infty, \qquad \operatorname{Re} g + \operatorname{Im} g \ge -kM(r). \tag{12}$$

In particular, if the real and imaginary parts of a and b are positive, $\operatorname{Re} g \ge -c$, $\operatorname{Im} g \ge c$, and $\alpha \le |a| \le M$, $\alpha \le |b| \le M$, then the operator L_0 is I-self-adjoint and therefore has no residual spectrum (^{1}). Now, using Theorem 2, it is easy to prove that the operator L_0 has a discrete spectrum if and only if the minimal operator M_0, generated by the differential expression

$$m[u] = -(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) + (\operatorname{Re} g + \operatorname{Im} g)u,$$

has a discrete spectrum. The operator M_0 has a discrete spectrum if and only if Molchanov' s condition (^{4}) is fulfilled. In the one-dimensional case it has the form

$$\int_x^{x+a} (\operatorname{Re} g + \operatorname{Im} g)\, dt \to \infty$$

as $x \to \infty$ and for every $a > 0$.

Using Theorem 1, it is easy to prove that the following holds:

Theorem 3. *Every I-self-adjoint extension of the minimal differential operator L_0, generated by the differential expression*

by the expression

$$l[y] = -\big[(p_1(x) + ip_2(x))y'\big]' + (g_1(x) + ig_2(x))y$$

in $L_2(0, \infty)$, has a discrete spectrum if at least one of conditions I–VI is satisfied:

I.

a) $p_1(x) \geq \alpha > 0$;

b) $g_1(x) \to +\infty$ as $x \to \infty$.

II.

a) $p_1(x) \geq \alpha > 0, \quad g_1(x) \geq -c$;

b)

$$\int_x^{x+a} g_1(t)\, dt \to +\infty \quad \text{as } x \to \infty \text{ and for any } a > 0.$$

III.

a) $p_1(x) > 0, \quad g_1(x) \geq -c$;

b)

$$\lim_{x\to\infty} x \int_x^{\infty} \frac{dt}{p_1(t)} = 0.$$

IV.

a) $p_1(x) > 0, \quad \displaystyle\int_0^{\infty} \frac{dt}{\sqrt{p_1(t)}} = c < \infty$;

b) $g_1(x) - {}^1/_{16}\, [p_1'(x)]^2 / p_1(x) \geq \alpha > 0$.

V.

a) $p_1(x) > 0, \quad \displaystyle\int_0^{\infty} \frac{dt}{\sqrt{p_1(t)}} = \infty$;

b) $g_1(x) - {}^1/_{16}\,[p_1'(x)]^2/p_1(x) \to +\infty$ as $x \to \infty$.

VI.

a) $p_1(x) > 0, \quad g_1(x) \geq -c;$

b) $g_1(x) + {}^1/_4\,[p''(x) - {}^1/_4[p'(x)]^2/p(x)] \to +\infty$ as $x \to \infty$.

The assertion of Theorem 3 holds if conditions I–VI are imposed either on p_2, g_2, or on $p_1 + p_2,\ g_1 + g_2$, or on $p_1 - p_2,\ g_1 - g_2$.

The author expresses his deep gratitude to M. A. Naimark for supervising the work and to R. S. Ismagilov for discussing the results.

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Received
10 IX 1969

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Note: Figure translations are in progress. See original paper for figures.