ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF INTEGRAL CONVOLUTION OPERATORS ON A FINITE INTERVAL WITH KERNELS WHOSE FOURIER TRANSFORMS ARE RATIONAL

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.43

MATHEMATICS

B. V. PAL' TSEV

ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF INTEGRAL CONVOLUTION OPERATORS ON A FINITE INTERVAL WITH KERNELS WHOSE FOURIER TRANSFORMS ARE RATIONAL

(Presented by Academician A. A. Dorodnitsyn on 11 III 1970)

Suppose that the kernel $k(s)$ of the integral equation

$$\int_0^T k(t-\tau)u(\tau)\,d\tau = \lambda_l u(t), \qquad 0 \le t \le T, \tag{1}$$

is the restriction to the segment $-T \le s \le T$ of some function defined on the entire real axis $-\infty < s < +\infty$. Denote by $\widetilde{K}(x)$ the Fourier transform of this function,

$$\widetilde{K}(x) = \int_{-\infty}^{+\infty} e^{-ixs} k(s)\,ds. \tag{2}$$

Works on the asymptotics of the eigenvalues of (1) as $l \to \infty$ for the most part concern the self-adjoint positive case (see, for example, ([2,3])). Thus, in ([3]), under the assumption of positivity, evenness, and monotone decrease of $\widetilde{K}(x)$ as $|x| \to \infty$ (slower than exponential), the formula $\lambda_l = \widetilde{K}(\pi l/T + o(l))$ was established. An analogous result in essence was obtained in ([2]). However, as far as the author knows, the non-self-adjoint case, i.e., when $\widetilde{K}(x)$ may be complex-valued, has so far scarcely been investigated.

Here we consider problem (1) for a very special class of kernels, on which, nevertheless, some interesting phenomena of the non-self-adjoint case may be observed. Suppose that

$$\widetilde{K}(x) = \frac{Q_m(x)}{P_n(x)} = \frac{(x-b_1)\cdots(x-b_m)}{(x-a_1^+)\cdots(x-a_q^+)(x-a_1^-)\cdots(x-a_p^-)}, \tag{3}$$

where $b_1, \dots, b_m$ are the roots of the polynomial $Q_m(x)$; $a_1^+, \dots, a_q^+$, $a_1^-, \dots, a_p^-$ are the roots of the polynomial $P_n(x)$ $(p+q=n)$, with $\operatorname{Im} a_j^+ > 0$ $(j = 1, \dots, q)$, $\operatorname{Im} a_j^- < 0$ $(j = 1, \dots, p)$; $b_k \neq a_j^\pm$. Equation (1) is an integral equation, therefore $n > m$.

Denote $r = n - m$ and, moreover, let

$$P^+(x) = (x - a_1^+) \cdots (x - a_q^+), \qquad P^-(x) = (x - a_1^-) \cdots (x - a_p^-).$$

Let

$$\widetilde{U}(x) = \frac{1}{2\pi} \int_0^T e^{-ixt} u(t)\, dt. \tag{4}$$

It is known $(^4)$ that if $\widetilde{K}(x)$ is of the form (3), then $\widetilde{U}(x)$, where $u(t)$ is a solution of (1), can be represented in the form

$$\widetilde{U}(x) = \frac{\widetilde{\Phi}^+(x) + e^{-ixT}\widetilde{\Phi}^-(x)}{\lambda - \widetilde{K}(x)} = \frac{\mu P_n(x)\,[\widetilde{\Phi}^+(x) + e^{-ixT}\widetilde{\Phi}^-(x)]}{P_n(x) - \mu Q_m(x)}, \tag{5}$$

where $\mu = 1/\lambda$. $\widetilde{\Phi}^+(x)$ and $\widetilde{\Phi}^-(x)$ are boundary values on the real axis of functions analytic in the upper and, respectively, in the lower half-planes of the complex z-plane, and they may be sought, for example, in the form

$$\widetilde{\Phi}^+(z) = \frac{c_p^+ z^{p-1} + c_{q-1}^+ z^{p-2} + \cdots + c_1^+}{P^-(z)}, \qquad \Phi^-(z) = \frac{c_q^- z^{q-1} + \cdots + c_1^-}{P^+(z)}, \tag{6}$$

where $c_1^+, \dots, c_p^+, c_1^-, \dots, c_q^-$ are certain constants.

Let us suppose for the time being that the roots $z_1, \dots, z_n$ of the polynomial $P_n(z) - \mu Q_m(z)$ are all simple and distinct from the roots of $P_n(z)$. Since $\widetilde{U}(x)$ is the boundary value on the real axis of an entire analytic function $\widetilde{U}(z)$ ($\widetilde{U}(x)$ is the Fourier transform of a finite function), the relations

$$\begin{gathered} \widetilde{\Phi}^+(z_1) + e^{-iz_1T}\widetilde{\Phi}^-(z_1) = 0, \\ \cdots \\ \widetilde{\Phi}^+(z_n) + e^{-iz_nT}\widetilde{\Phi}^-(z_n) = 0. \end{gathered} \tag{7}$$

must be satisfied. (7) is a homogeneous system of n linear equations with respect to the n unknowns $c_1^+, \dots, c_p^+, c_1^-, \dots, c_1^-$. Introduce the notation

$$\Delta_{s,t}(z_1,\dots,z_k)=\begin{vmatrix} \dfrac{z_1^{s-1}}{P^-(z_1)} & \cdots & \dfrac{1}{P^-(z_1)} & \dfrac{z_1^{t-1}e^{-iz_1T}}{P^+(z_1)} & \cdots & \dfrac{e^{-iz_1T}}{P^+(z_1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{z_k^{s-1}}{P^-(z_k)} & \cdots & \dfrac{1}{P^-(z_k)} & \dfrac{z_k^{t-1}e^{-iz_kT}}{P^+(z_k)} & \cdots & \dfrac{e^{-iz_kT}}{P^+(z_k)} \end{vmatrix}, \qquad (8)$$

$s+t=k$. The condition for the existence of a nontrivial solution of the system (7) is written in the form

$$\Delta_{p,q}(z_1,\dots,z_n)=0. \qquad (9)$$

(9) is the characteristic equation for determining μ_l. Since $\lambda_l\to 0$ as $l\to\infty$, and $\mu_l=1/\lambda_l\to\infty$, it is necessary to find the asymptotic behavior of the roots $z_1,\dots,z_n$ as $\mu\to\infty$. It is not difficult to obtain the following assertion:

The polynomial $P_n(z)-\mu Q_m(z)$ has $r=n-m$ roots $z_1,\dots,z_r$ tending to ∞ and admitting, for sufficiently large $|\mu|$, the expansion

$$z=\mu^{1/r}-\frac{P_1-Q_1}{r}+\sum_{j=1}^{\infty}\frac{c_j}{(\mu^{1/r})^j}, \qquad (10)$$

where

$$P_1=-\left(\sum a_j^+ +\sum a_j^-\right),\qquad Q_1=-\left(\sum b_j\right), \qquad (11)$$

and m roots $z_{r+1},\dots,z_n$ which tend to $b_1,\dots,b_m$; moreover, if b is a root of $Q_m(z)$ of multiplicity k, then among $z_{r+1},\dots,z_n$ there exist k distinct roots which admit, as $|\mu|\to\infty$, the expansion

$$z=b+\left(\frac{P_n(b)\,k!}{Q_m^{(k)}(b)}\right)^{1/k}\frac{1}{\mu^{1/k}}+\sum_{j=2}^{\infty}\frac{\tilde{c}_j}{(\mu^{1/k})^j}, \qquad (12)$$

$$Q_m^{(k)}(z)=\frac{d^k}{dz^k}Q_m(z).$$

Hence it is clear that, for sufficiently large $|\mu|$, the roots $z_1,\dots,z_n$ will be simple and distinct from the roots of $P_n(z)$.

Using the expansions (10) and (12), and considering (9) in different sectors of the complex variable $\rho=\mu^{1/r}$, we arrive at the following result:

There are two essentially different cases, depending on the relation between p, q, and m.

I. $|p-q|>m$. For all λ_l there is a single asymptotic expansion

$$\lambda_l \approx \exp\left[-i\frac{\pi}{2}(|p-q|-m)\operatorname{sign}(p-q)\right]\left(\frac{\pi l}{T\sin[\pi\min(p,q)/r]}\right)^{-r}\times\left[1+\frac{\theta_1}{l}+\sum_{k=2}^{\infty}\frac{Q_k}{l^k}\right] \tag{13}$$

$$(l=1,2,3,\ldots),\quad \text{where}\quad \theta_1=-[(\min(p,q))^2+r]/2.$$

Before passing to case II, let us introduce one more notation. Let $z_1(\mu),\ldots,z_m(\mu)$ be m complex-valued functions of the parameter μ, with $z_1(\mu)\to b_1,\ldots,z_m(\mu)\to b_m$ as $\mu\to\infty$. Moreover, for sufficiently large $|\mu|$, all $z_1,\ldots,z_m$ are pairwise distinct, although some among $b_1,\ldots,b_m$ may coincide with one another. Denote

$$\frac{\Delta_{s,t}(b_1,\ldots,b_m)}{W_m(b_1,\ldots,b_m)}=\lim_{\mu\to\infty}\frac{\Delta_{s,t}(z_1(\mu),\ldots,z_m(\mu))}{W_m(z_1(\mu),\ldots,z_m(\mu))}, \tag{14}$$

where $W_m(z_1,\ldots,z_m)$ is the Vandermonde determinant of the numbers $z_1,\ldots,z_m$. Obviously, such a limit always exists and depends only on $b_1,\ldots,b_m$, but not on the form of $z_1(\mu),\ldots,z_m(\mu)$. It is not difficult to give an expression for (14) directly in terms of $b_1,\ldots,b_m$ and the roots of the polynomial $P_n(x)$.

In case II we impose on $\widetilde{K}(x)$ some additional conditions of algebraic character —the "regularity" conditions R).

II. $|p-q|\le m$. Consider two subcases:

a) $r=n-m$ is odd. Suppose that the conditions R) are also satisfied:

$$A=\frac{\Delta_{p-(r-1)/2,\;q-(r+1)/2}(b_1,\ldots,b_m)}{W_m(b_1,\ldots,b_m)}\neq 0,$$

$$B=\frac{\Delta_{p-(r+1)/2,\;q-(r-1)/2}(b_1,\ldots,b_m)}{W_m(b_1,\ldots,b_m)}\neq 0. \tag{15}$$

Then the eigenvalues λ_l can be split into two series λ_l^+ and λ_l^-, each of which can be represented by an asymptotic series

$$\lambda_l^\pm \approx \pm\left(\frac{2\pi l}{T}\right)^{-r}\left[1+\frac{\theta_1^\pm}{l}+\sum_{k=2}^{\infty}\frac{\theta_k^\pm}{l^k}\right]\qquad(l=1,2,3,\ldots), \tag{16}$$

$$\theta_1^{\pm} = \mp \left[\frac{T}{2\pi}(P_1 - Q_1) \pm \frac{(r-1)^2}{8} - \frac{r}{4}(p - q + m - 1)\,\mathrm{sign}(p-q) + \frac{ir}{2\pi}\ln\frac{B}{A} \right],$$

where P_1 and Q_1 are given by (11).

b) r is even. The "regularity" condition has the form

$$\text{R)} \qquad A = \frac{\Delta_{p-r/2,\, q-r/2}(b_1, \dots, b_m)}{W_m(b_1, \dots, b_m)} \neq 0. \tag{17}$$

Then the eigenvalues λ_l can likewise be split into two series λ_l^+ and λ_l^-, for each of which the asymptotic expansions

$$\lambda_l^{\pm} \approx \left(\frac{2\pi l}{T}\right)^{-r} \left[1 + \frac{\theta_1^{\pm}}{l} + \sum_{k=2}^{\infty} \frac{\theta_k^{\pm}}{l^k}\right] \qquad (l = 1, 2, 3, \dots),$$

$$\theta_1^{\pm} = -\frac{r}{4}\left(\frac{r}{2} \pm 1\right). \tag{18}$$

hold.

The eigenvalues in all cases, starting with sufficiently large l, are simple. The series (13), (16), and (18) are understood as asymptotic and, generally speaking, are not convergent.

In conclusion we make several remarks. If in case II the conditions R) are not satisfied, then formulas (16) or (18) become invalid. It may then happen that for λ_l there do not exist asymptotic series of the form (16) or (18) at all. We note that in case I no conditions of type R) are imposed.

From (16) and (18) it is clear that in case II the first term of the asymptotics of λ_l coincides with the expression obtained in (^{3}). Case II may therefore be regarded as "weakly" non-self-adjoint. As for case I, the expression for the first term of the asymptotics is already given by a formula different from the formula for the self-adjoint case. For example, if one sets $p = q + 1$, $m = 0$, then

$$\lambda_l \sim e^{-i\pi/2} \left(\pi l/T \sin\frac{\pi q}{2q+1}\right)^{-(2q+1)}$$

and, despite the fact that the first term in the expansion of $\widetilde{K}(x)$ as $x \to \infty$ is real, the eigenvalues are asymptotically imaginary. Moreover, if one sets, for example, $q = 0$, $p = n$, then (1) has no eigenvalues at all, since the corresponding integral operator will be Volterra. Thus, case I has no analogue for self-adjoint operators and is, in a certain sense, intermediate between the Volterra and "weakly" non-self-adjoint cases.

Finally, we note that the results obtained are a generalization of the results of ([1]) for ordinary differential operators to the case of convolution integral operators.

Computing Center
Academy of Sciences of the USSR
Moscow

Received
11 III 1970

CITED LITERATURE

[1] G. D. Birkhoff, Trans. Am. Math. Soc., **9**, 373 (1908).
[2] M. Rosenblatt, J. Math. Mech., **12**, 4, 619 (1963).
[3] H. Widom, Arch. Rational. Mech. Analys., **17**, 3, 215 (1964).
[4] M. P. Tanin, Izv. vyssh. uchebn. zaved., Matematika, **2**, 31 (1963).

Note: Figure translations are in progress. See original paper for figures.