ON HEREDITARY AND COLLECTIVE NORMALITY OF A PERIPHERALLY BICOMPACT TREE-LIKE SPACE

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ON HEREDITARY AND COLLECTIVE NORMALITY OF A PERIPHERALLY BICOMPACT TREE-LIKE SPACE

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As numerous and simple examples show, a peripherally bicompact Hausdorff space need not be normal. Such a space, as Freudenthal showed (^1), is only automatically completely regular. However, a tree-like* peripherally bicompact space is normal, as is proved here. Moreover, the following theorems are valid.

Theorem 1. *Any two disjoint closed sets of a tree-like peripherally bicompact space are separated by a discrete set.*

Proof. In other words, it is necessary to prove that if $A, B \subset X$ are closed, where X is peripherally bicompact and tree-like, $A \cap B = \Lambda$, then there exists a discrete set $C \subset X$ such that $X \setminus C = X_1 \cup X_2$, $A \subset X_1$, $B \subset X_2$, where X_1 and X_2 are open in X, $X_1 \cap X_2 = \Lambda$.

Any two points in a tree-like peripherally bicompact space are joined by a unique ordered continuum (^2). If $D \subset V \subseteq X$, then by the linear hull of the set D in the set V we shall mean the union of all those ordered continua which join points of D pairwise and are entirely contained in V.

Lemma 1. *Let $D \subset V \subseteq X$, where D is closed, V is open, and X is a tree-like peripherally bicompact space. Then the linear hull LD of the set D in the set V is closed in X.*

We prove the lemma. Suppose that $x \in [LD] \setminus LD$. Take a connected neighborhood Ox with finite boundary such that $Ox \cap D = \Lambda$. Since $x \in [LD] \setminus LD$, there exists an infinite number of ordered continua entirely belonging to $LD \cap [Ox]$ and joining pairwise boundary points of Ox. But the boundary of Ox is finite, and among these continua two distinct ones will be found joining a pair of boundary points of Ox, which contradicts Lemma 1 of (^2).

Lemma 2. *The boundary of a connected open subset in a tree-like peripherally bicompact space is punctiform.*

Lemma 3. *If a point c separates a connected subset D of a tree-like peripherally bicompact space X between points a and b, $a, b \in D$, then the point c also* *separates any subset containing D between the same points.*

Returning to the proof of Theorem 1, take the linear hull LA of the set A in the set $X \setminus B$ and then the linear hull LB of the set B in $X \setminus LA$. We decompose the locally connected set $\widetilde{X} = X \setminus (LA \cup LB)$ into open components: $\widetilde{X} = \bigcup_\alpha \widetilde{X}_\alpha$. It is further asserted that every $[\widetilde{X}_\alpha]$ contains not more than one point of LA and LB.

Indeed, if, for example, $a_1, a_2 \in [\widetilde{X}] \cap LA$, then two possibilities arise. The first: the ordered continuum L, joining the points a_1 and a_2, and lying entirely in $[\widetilde{X}_\alpha]$, such that $L \cap LB = \Lambda$. But then there is a contradiction with Lemma 2. Second: suppose there is a point $b \in L \cap LB$. But then, by Lemma 3, the point b separates $[\widetilde{X}_\alpha]$ between a_1 and a_2, and also, as is easy to conclude, b separates $\widetilde{X}_\alpha$, which contradicts the connectedness of $\widetilde{X}_\alpha$.

Next, if $[\widetilde{X}_\alpha] \setminus \widetilde{X}_\alpha$ consists of two points, then we take a point $c_\alpha \in \widetilde{X}_\alpha$ separating $[\widetilde{X}_\alpha]$ between these two points. The union of all such points c_α forms a discrete set C closed in X.

The locally connected $X \setminus C$ is decomposed into open components; those of them which intersect LA are united—we obtain OA; those which intersect LB are united—we obtain OB. There is no component which would intersect both LA and LB. Suppose the contrary: suppose there is a component P of the set $X \setminus C$ such that $P \cap LA \neq \Lambda$ and $P \cap LB \neq \Lambda$. But then there is a component Q of the set $P \setminus (LA \cup LB)$ such that $[Q]$ intersects both LA and LB, in view of the connectedness of P. There is a component $\widetilde{X}_\alpha$ of the set $X \setminus (LA \cup LB)$ containing Q. From the latter it follows that in the set C there is a point c_α separating $[Q]$ between the points

$$a = [Q] \cap LA$$

and

$$b = [Q] \cap LB,$$

and moreover $c_\alpha \in Q$, but this gives a contradiction. Thus the theorem is proved.

Corollary 1. *The large inductive dimension of a dendroidal peripherally bicompact space is equal to one.**

In addition, Theorem 1 implies the normality of a predendroidal peripherally bicompact space. But a stronger assertion is true.

Theorem 2. *A dendroidal peripherally bicompact space is hereditarily normal.*

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* A space is tree-like if it is connected and if any two points in it are separated by a third. It is automatically Hausdorff.

Proof. Let $\widetilde{X} \subseteq X$ be a subspace of the dendroidal peripherally bicompact space X, and let $A, B \subset \widetilde{X}$ be closed in $\widetilde{X}$ and $A \cap B = \Lambda$. Consider the set $C = [A] \cap [B]$. Obviously, $C \subseteq X \setminus \widetilde{X}$. We decompose the locally connected open set $X \setminus C$ in X into open components:

$$X \setminus C = \bigcup_{\alpha} X_\alpha.$$

Each X_α is a peripherally bicompact dendroidal subspace of X. Indeed, every subset of a peripherally bicompact dendroidal space is itself peripherally bicompact, since in a peripherally bicompact dendroidal space every point has arbitrarily small neighborhoods with finite boundaries (^{3}). Therefore, by Theorem 1, for the sets $A_\alpha = A' \cap X_\alpha$ and $B_\alpha = B' \cap X_\alpha$, where $A' = [A] \setminus C$ and $B' = [B] \setminus C$, there exist disjoint neighborhoods OA_α and OB_α such that $OA_\alpha, OB_\alpha \subset X_\alpha$. Let us also note that $A \subseteq A'$ and $B \subseteq B'$.

It is not difficult to see that

$$OA = \bigcup_{\alpha} OA_\alpha$$

and

$$OB = \bigcup_{\alpha} OB_\alpha$$

are disjoint neighborhoods of the sets A and B. This means that $\widetilde{X}$ is normal, and Theorem 2 is proved.

Corollary 2. *A dyadic dendroidal bicompactum is metrizable.*

We had put forward a hypothesis on the metrizability of a hereditarily normal dyadic bicompactum, which B. A. Efimov succeeded in proving (^{4}). In combination with Theorem 2 this gives Corollary 2.

Theorem 3. *A dendroidal peripherally bicompact space is collectively normal.*

Lemma 4. *The set LD mentioned in Lemma 1 is locally connected.*

Let us prove the lemma. Let $x \in LD$ be a point and let Ox be its connected neighborhood in X such that $Ox \subset V$. It is asserted that the set $O_1x = Ox \cap LD$, which is a neighborhood of the point x in the set LD, is connected. Indeed, every point $y \in O_1x$ can be joined to the point x by an ordered continuum lying entirely in Ox, and hence also lying in O_1x in the sense of the linear hull. The lemma is proved.

* The question of Ind of a dendroidal peripherally bicompact space was raised by B. A. Pasynkov.

Lemma 5. Let $\Gamma \subset X$ be a discrete subset of the tree-like peripherally bicompact space X, and suppose that $X \setminus \Gamma$ is connected. Then there exists a closed set $C \subset X$ such that $X \setminus C$ is decomposed into open components, each of which contains at most one point of Γ.

We prove the lemma. Let $\Gamma = \{x_\alpha\}$; for an arbitrary $x_\alpha \in \Gamma$ denote $\Gamma_\alpha = \Gamma \setminus x_\alpha$. The linear hull $L\Gamma_\alpha$ of the set Γ_α in X is closed by Lemma 1, is connected, and $x_\alpha \notin L\Gamma_\alpha$. The latter follows from the fact that $X \setminus x_\alpha$ is connected by the hypothesis of the lemma, and any two points in $X \setminus x_\alpha$ are joined by a unique ordered continuum (see Lemma 1 from (^{2})). By V_α we denote the open component of the set $X \setminus L\Gamma_\alpha$ containing the point x_α. It is asserted that $V_{\alpha_1} \cap V_{\alpha_2} = \Lambda$. Suppose the contrary: there is a point $c \in V_{\alpha_1} \cap V_{\alpha_2}$. Then $V_{\alpha_1} \cup V_{\alpha_2}$ is connected. We note that $[V_{\alpha_1}] \setminus V_{\alpha_1}$ consists of a single point; denote it by y_{α_1}. This latter circumstance can be justified with the help of Lemma 3. Correspondingly, by y_{α_2} denote the point $[V_{\alpha_2}] \setminus V_{\alpha_2}$. From what has been said it is clear that

$$[V_{\alpha_1} \cup V_{\alpha_2}] \setminus (V_{\alpha_1} \cup V_{\alpha_2}) = y_{\alpha_1} \cup y_{\alpha_2}$$

and $V = [V_{\alpha_1} \cup V_{\alpha_2}]$ is a tree-like peripherally bicompact subspace of the space X. We note that $y_{\alpha_2} \notin V_{\alpha_1}$. Indeed, if $y_{\alpha_2} \in V_{\alpha_1}$, then we would join x_{α_2} with y_{α_1} by an ordered continuum, which necessarily would pass through y_{α_2}, and then join the point y_{α_1} with $x_\beta \in \Gamma \setminus (x_{\alpha_1}, x_{\alpha_2})$; consequently, we would obtain that x_{α_2} and x_β are joined by an ordered continuum to which the point y_{α_2} belongs, which means: $y_{\alpha_2} \in L\Gamma_{\alpha_1}$. But the latter at the same time means that $y_{\alpha_2} \notin V_{\alpha_1}$. Accordingly, $y_{\alpha_1} \notin V_{\alpha_2}$. However, the point $y_{\alpha_1} \in L\Gamma_{\alpha_1}$ and the point $y_{\alpha_2} \in L\Gamma_{\alpha_1}$, for, joining x_{α_2} and x_β by an ordered continuum, we see that it passes through y_{α_2}. The ordered continuum L_1, joining x_β with y_{α_1}, does not intersect V_{α_1}, since $x_\beta, y_{\alpha_1} \in L\Gamma_{\alpha_1}$, and, moreover, $L_1 \cap V_{\alpha_2} = \Lambda$, since the continuum joining x_β with x_{α_1} belongs entirely to $L\Gamma_{\alpha_2}$ and contains L_1 in itself. Thus,

$$L_1 \cap (V_{\alpha_1} \cup V_{\alpha_2}) = \Lambda.$$

Similarly,

$$L_2 \cap (V_{\alpha_1} \cup V_{\alpha_2}) = \Lambda,$$

where L_2 is the ordered continuum joining the points x_β and y_{α_2}. Denote by L the connected closed set $L_1 \cup L_2$. It is seen that the connected closed sets L and $[V]$ intersect in two points y_{α_1} and y_{α_2}, which contradicts the fact that in a tree-like peripherally bicompact space the intersection of connected closed sets is empty or connected (see Lemma 2 from (^{2})). Hence, $V_{\alpha_1} \cap V_{\alpha_2} = \Lambda$, and as the set C we take $X \setminus \bigcup V_\alpha$. Lemma 5 is proved.

Proof of Theorem 3. Let $\{A_\alpha\}$ be a discrete family of closed subsets of the tree-like peripherally bicompact space X. By transfinite induction we extend each A_α to some closed locally connected $\widetilde{A}_\alpha$ in the following way. The set

$$\widetilde{A}_1 = LA_1$$

in the set

$$X \setminus \bigcup_{\alpha \neq 1} A_\alpha.$$

If all $\widetilde{A}_\alpha$ have been constructed for all $\alpha < \beta$, then $\widetilde{A}_\beta$ is defined as follows: $\widetilde{A}_\beta$ is the union of all those ordered continua joining pairs of points of A_β which do not intersect the closed set

$$\bigcup_{\alpha \leq \beta} \widetilde{A}_\alpha \cup \bigcup_{\alpha \neq \beta} A_\alpha.$$

In order that the inclusion $A_\alpha \subset \widetilde{A}_\alpha$ hold, by definition we adjoin to $\widetilde{A}_\alpha$ all points of A_α itself.

By Lemmas 1 and 4, each $\widetilde{A}_\alpha$ is a locally connected closed set in X. Decompose $\widetilde{A}_\alpha$ into components; we obtain a discrete system of closed sets:

$$\widetilde{A}_\alpha = \bigcup_\gamma \widetilde{A}_{\alpha\gamma}.$$

The system $\{\widetilde{A}_{\alpha\gamma}\}$, over all admissible α and γ, is a discrete system of connected closed sets.

The subspace

$$\widetilde{X} = X \setminus \bigcup_{\alpha,\gamma} \widetilde{A}_{\alpha\gamma}$$

is decomposed into open components:

$$\widetilde{X} = \bigcup_\beta \widetilde{X}_\beta.$$

At the same time, $[\widetilde{X}_\beta]$ for any β contains at most one point from each $\widetilde{A}_{\alpha\gamma}$; this means that

$$\Gamma_\beta = [\widetilde{X}_\beta] \setminus \widetilde{X}_\beta$$

is a discrete set. Applying Lemma 5 to the subspace $[\widetilde{X}_\beta]$ and its discrete subset Γ_β, we obtain a closed set C_β, $C_\beta \subset \widetilde{X}_\beta$. It is possible to show that the set $C = \bigcup_\beta C_\beta$ is closed. Suppose the contrary: there is a point $x \in [C] \setminus C$. Take an arbitrary neighborhood O_x of it with finite boundary. The neighborhood O_x intersects infinitely many sets of the form $\widetilde{X}_\beta$, but since the boundary of O_x is finite and the family $\{\widetilde{X}_\beta\}$ is disjoint, this means that infinitely many sets of the form $\widetilde{X}_\beta$ are entirely contained in O_x. Thus every neighborhood of the point x with finite boundary contains infinitely many sets of the form $\widetilde{X}_\beta$ and, consequently, intersects infinitely many sets of the form $\widetilde{A}_\alpha$, which contradicts the discreteness of the family $\{\widetilde{A}_\alpha\}$. Hence C is closed. After this it is not hard to see that $X \setminus C$ decomposes into open components, each of which contains at most one set of the form $\widetilde{A}_\alpha$; from this it follows at once that the discrete family $\{A_\alpha\}$ is placed in a system of pairwise disjoint neighborhoods. Consequently, X is collectively normal. The theorem is proved.

Example. There exist dendroid spaces that are not normal, as the following example shows*. On the half-plane we introduce a new topology as follows. If $x \in L$, where L is the boundary line, then a basic neighborhood O_x is an open disk tangent to L at the point x, together with the point x. If, however, $x \notin L$, then its basic neighborhood is an arbitrary interval containing x and perpendicular to L. The space obtained in this way is dendroid, completely regular, but not normal (compare it with the Niemytzki space!).

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* The example answers a question raised by O. V. Lokutsievskii.

Note: Figure translations are in progress. See original paper for figures.

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