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MATHEMATICS

1970

SovietRxiv

Abstract

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UDC 517.513

MATHEMATICS

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ON THE BOUNDEDNESS OF PSEUDODIFFERENTIAL OPERATORS IN WEIGHTED SPACES

(Presented by Academician V. I. Smirnov on 28 V 1969)

1°. Let R^n be n-dimensional Euclidean space of points $x = (x_1, \ldots$

$$\ldots, x_n); \quad \rho^2 = |x|^2 = \sum_{i=1}^{n} x_i^2; \quad S = \{x: \ |x| = 1\}; \quad \theta = x\rho^{-1} \in S.$$

If $\omega = (\omega_1, \ldots, \omega_n)$ is a set of nonnegative integers (a multi-index), then

$$D_x^\omega = D_{x_1}^{\omega_1} \ldots D_{x_n}^{\omega_n}; \qquad D_{x_i} = \partial/\partial x_i; \qquad x^\omega = x_1^{\omega_1} \ldots x_n^{\omega_n};$$

$$|\omega| = \sum_{i=1}^{n} \omega_i; \qquad \omega! = \omega_1! \ldots \omega_n!$$

Denote by D the space of infinitely differentiable functions with compact supports. We define the space $L_{p,\alpha}$ as the set of functions defined on R^n with finite norm

$$\|u\|_{L_{p,\alpha}} = \|\rho^\alpha u\|_{L_p} < \infty, \qquad 1 < p < \infty.$$

We shall specify a singular integral operator with symbol $\Phi(x, \xi)$ in the form

$$Au = F_{\xi \to x}^{-1} \Phi(x, \xi) F_{y \to \xi} u, \tag{1}$$

where $F_{y \to \xi} u$ is the Fourier transform of the function $u \in D$. We shall also use the representation of a singular operator by means of a kernel, namely:

$$Au = \int_{R^n} K(x, x - y) u(y)\, dy, \tag{1'}$$

where $K(x, x-y) = f(x, \theta_{xy})|x-y|^{-n}$, $\theta_{xy} = (x-y)|x-y|^{-1}$.

The boundedness of the one-dimensional singular integral in the space $L_{p,\alpha}$ $(-1/p < \alpha < 1/p';\ 1/p + 1/p' = 1)$ was proved by K. I. Babenko ([2]). E. M. Stein ([3]) obtained a theorem on the boundedness of the operator (1′) in the space $L_{p,\alpha}$ for arbitrary n and $\alpha \in (-n/p, n/p')$ under the condition that $\operatorname{vrai\,sup}|f(x,\theta)| < \infty$. B. A. Plamenevskii ([4]) and the author ([5]) independently proved that, for $\alpha \in (-n/2, n/2)$, $|\alpha| \neq n/2 + k$ $(k = 0, 1, \ldots)$, the operator (1) with symbol $\Phi(\xi)$, defined on certain dense subsets in $L_{2,\alpha}$, is bounded in the space $L_{2,\alpha}$.

The present paper is devoted to generalizing these results in various directions.

Denote by Q_s the set of functions $u \in D$ satisfying the conditions

$$\int_{R^n} x^\omega u(x)\, dx = 0,$$

where ω is any multi-index of order $0 \le |\omega| \le s$. By Q_{-s} we denote the set of functions $u \in D$ such that $0 \notin \operatorname{supp} u$, and satisfying

conditions

$$\int_0^\infty \rho^{-q} u(\rho, \theta)\, d\rho = 0, \qquad q = 1, 2, \ldots,$$

The sets Q_s and Q_{-s} are dense in the space $L_{p,\alpha}$.

Theorem 1.1. I. Let $n/p < \alpha < n/p'$ and

$$\int_S |f(x,\theta)|^{p'}\, d\theta \leqslant \text{const} < \infty. \tag{2}$$

Then the operator (1′) is bounded in the space $L_{p,\alpha}$.

II. Let $\alpha > n/p'$, $\alpha \neq n/p' + k$ $(k = 1, 2, \ldots)$, condition (2) be satisfied, and

$$\int_{R \le |x| \le 2R} |D_z^\omega K(x,z)|_{z=x}|^p\, dx \leqslant \text{const} \cdot R^{-n(p-1)-(\omega)p},$$

where ω is any multi-index of order $0 \le |\omega| \le s = [\alpha - n/p']^*$ and $R \in (0, \infty)$. Suppose further that for some $\varkappa \in (\alpha - n/p', s+1]$ the inequality

$$\int_{R \le |x| \le 2R} |K^+(x,y)|^p\, dx \leqslant \text{const} \cdot R^{-n(p-1)} \left(\frac{|y|}{R}\right)^{\varkappa p},$$

is satisfied, where

$$K^{+}(x,y)=K(x,x-y)-\sum_{|\omega|=0}^{s} D_z^{\omega}K(x,z)|_{z=x}\frac{y^{\omega}}{\omega!}.$$

Then the operator $(1')$, defined on the set $Q(s)$, is bounded in the space $L_{p,\alpha}$.

III. Let $\alpha<-n/p$, $|\alpha|\neq n/p+k$ $(k=1,2,...)$, condition (2) be satisfied, and

$$\int_S |D_x^{\omega}K(x,x-y)|_{x=0}|^{p'}\,d\theta\leqslant \mathrm{const}\cdot|y|^{-(n+|\omega|)p'},$$

where ω is a multi-index of order $0\leq|\omega|\leq s=[|\alpha|-n/p]$. Suppose further that for $|y|>2|x|$ and $\varkappa\in(|\alpha|-n/p,s+1]$ the inequality

$$\int_S |K^{-}(x,y)|^{p'}\,d\theta\leqslant \mathrm{const}\cdot|y|^{-np'}\left(\frac{|x|}{|y|}\right)^{\varkappa p'},$$

is satisfied, where

$$K^{-}(x,y)=K(x,x-y)-\sum_{|\omega|=0}^{s} D_x^{\omega}K(x,x-y)|_{x=0}\frac{x^{\omega}}{\omega!}.$$

Then the operator $(1')$, defined on the set Q_{-s}, is bounded in the space $L_{p,\alpha}$.

For $\alpha>n/p'$ the operator $(1')$, defined on the set Q_s, $s=[\alpha-n/p']$, coincides with the operator

$$A^{+}u=\int_{R^n}K^{+}(x,y)u(y)\,dy.$$

For $\alpha<-n/p$ the operator $(1')$, defined on the set Q_{-s}, $s=[|\alpha|-n/p]$, coincides with the operator

$$A^{-}u=\int_{R^n}K^{-}(x,y)u(y)\,dy.$$

2°. Let us consider separately the case of the space $L_{2,\alpha}$. By definition, $\Phi(x,\xi)\in L_{\infty}W_2^l(S)$ if $\Phi(x,\xi)$ belongs to the space $W_2^l(S)$

* Here and below $[r]$ denotes the integer part of the number r.

(the space of S. L. Sobolev–L. N. Slobodetskii on the sphere S) with respect to the variable ξ and

$$\operatorname*{vrai\,sup}_{x\in R^n}\|\Phi(x,\xi)\|_{W_2^l(s)}<\infty.$$

Theorem 2. Let $\Phi(x,\xi) \in L_\infty W_2^l(S)$, $l > (n-1)/2 + |\alpha|$, $\alpha \neq n/2 + k$ $(k = 0, 1, \ldots)$. Then the operator (1), defined on the whole space $L_{2,\alpha}$ for $|\alpha| < n/2$ and on the set $Q_s(Q_{-s})$ for $|\alpha| > n/2$ $(s = [|\alpha| - n/2])$, is bounded in the space $L_{2,\alpha}$.

The proof is based on the following estimate

$$\sum_{m=0}^{m_k} \left\| F^{-1}_{\xi\to x} Y_{mk}(\xi) F_{y\to\xi} u \right\|^2_{L_{2,\alpha}} \le \mathrm{const} \cdot k^{n-2+2|\alpha|} \|u\|^2_{L_{2,\alpha}},$$

where $Y_{mk}(\xi)$ is a spherical function of order k.

For $\alpha = 0$, theorem 2 becomes the theorem of S. G. Mikhlin (1)—M. S. Agranovich (7). There is an assertion analogous to theorem 2 in the spaces L_2 with weight function $\lambda(x) = (1 + |x|)^\alpha$ and with weight function $\lambda(x)$ equal to $|x|^\alpha$ for $|x| \le 1$ and to unity for $|x| > 1$.

3°. Let H^μ be the function space defined in (6), endowed with the norm

$$\|u\|_{H^\mu} = \left(\int_{R^n} \mu^2(\xi) \left| F_{x\to\xi} u \right|^2 d\xi \right)^{1/2} < \infty.$$

We shall say that $a(x,\xi) \in H^\mu L_\infty$ if $a(x,\xi)$ belongs to the space H^μ with respect to the variable x and

$$\operatorname*{vrai\,sup}_{\xi\in R^n} \|a(\cdot,\xi)\|_{H^\mu} < \infty.$$

The class of functions $L_\infty H^\mu$ is defined analogously.

We define the space $L_2(\mu)$ as the set of functions given on R^n for which the norm

$$\|u\|_{L_2(\mu)} = \left(\int_{R^n} |\mu(x) u(x)|^2 \, dx \right)^{1/2} < \infty$$

is finite.

Theorem 3. The pseudodifferential operator

$$Bu = F^{-1}_{\xi\to x} b(x,\xi) F_{y\to\xi} u \tag{3}$$

is bounded in the space L_2 if $b(x,\xi) \in H^\mu L_\infty$ and

$$\int_{R^n} \mu^{-2}(\tau)\, d\tau < \infty. \tag{4}$$

Condition (4) is, in a certain sense, sharp. Namely, for any function μ not satisfying condition (4), one can find a symbol $b(x) \in H^{\mu}$ such that the operator (3) is not bounded in L_2. A close result for singular operators, refining the theorem of S. G. Mikhlin—M. S. Agranovich mentioned above, was obtained in ([11]).

Theorem 4. Let a pseudodifferential operator with symbol $b(x,\xi)$ be bounded in the space $L_2(\nu)$, and let the function $\varphi(x,\xi)$ belong to $L_\infty H^{\mu}$.

Then the pseudodifferential operator with symbol $\varphi(x,\xi)b(x,\xi)$ is bounded in the space $L_2(\nu)$, if

$$\int_{R^n} \nu^2(x+y)\mu^{-2}(y)\,dy \le \mathrm{const}\cdot\nu^2(x).$$

The fact that the continuity conditions for pseudodifferential operators in L_2 can at the same time be multiplier conditions was first noted in the works of M. Sh. Birman and M. Z. Solomyak ([9,10]).

Theorem 3 refines, and Theorem 4 generalizes, analogous assertions from works ([9], [10]), obtained by methods of the theory of double operator integrals.

4°. By $L_{p,\alpha,\beta,\gamma}$ we denote the completion of the space of functions $u \in D$ $(0 \notin \operatorname{supp} u)$ with respect to the norm

$$\|u\|_{L_{p;\alpha,\beta,\gamma}} = \|\rho^{\alpha}(-\Delta)^{\beta/2}(-\delta)^{\gamma/2}u\|_{L_p},$$

where Δ is the Laplace operator, δ is the spherical part of Δ, and α,β,γ are real numbers.

Let us formulate some properties of the spaces $L_{p,\alpha,\beta,\gamma}$.

1) Let $-n/p < \alpha - \beta < \alpha < n/p'$. Then, for functions $u \in D$, the inequalities

$$c_1\|\rho^{\alpha}(-\Delta)^{\beta/2}u\|_{L_p} \le \|(-\Delta)^{\beta/2}\rho^{\alpha}u\|_{L_p} \le c_2\|\rho^{\alpha}(-\Delta)^{\beta/2}u\|_{L_p},$$

hold, where c_1, c_2 are constants independent of the function u.

2) Let $0 < \beta \ne n/p + k,\ k = 0, 1, \ldots.$ Then for all functions $u \in D$ $(0 \notin \operatorname{supp} u)$ the inequalities

$$c_1\|(-\Delta)^{\beta/2}u\|_{L_p} \le \|(-\Delta_\rho)^{\beta/2}u\|_{L_p} + \|\rho^{-\beta}(-\delta)^{\beta/2}u\|_{L_p} \le c_2\|(-\Delta)^{\beta/2}u\|_{L_p}, \quad (5)$$

are valid, where Δ_ρ denotes the radial part of the Laplace operator.

In the proof of inequality (5) the estimate

$$\|\rho^{-\beta}u\|_{L_p} \leq c\|(-\Delta)^{\beta/2}u\|_{L_p},$$

is used, where $0 < \beta \neq n/p + k,\ k = 0, 1, ...$, previously proved by V. P. Il' in ([8]) for $0 < \beta < n/p$.

With the aid of the theorem from 1° and the formulated properties of the spaces $L_{p,\alpha,\beta,\gamma}$, one proves theorems on the boundedness of operators whose symbols, for large $|\xi|$, have the form

$$\sigma(x, \xi) = \sum_{k=1}^{N} |\xi|^{\lambda_k} \sigma_k(x, \xi),$$

where λ_k are real numbers, and $\sigma_k(x, \xi)$ are functions positively homogeneous of degree zero, sufficiently smooth and, for large $|x|$, independent of x.

The author expresses sincere gratitude to B. G. Mazy' a for proposing the problem and for assistance in carrying out the work.

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Received
22 V 1969

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Note: Figure translations are in progress. See original paper for figures.