ON STONE' S THEOREM

1970

SovietRxiv

Abstract

Full Text

UDC 519.48+513.83

V. V. PASHENKOV

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(Presented by Academician P. S. Aleksandrov on 17 III 1970)

We shall agree to understand by the word *space* a Hausdorff topological space possessing a prebase of open-and-closed sets. By a topological algebra B we shall understand a Boolean algebra which is at the same time a space whose topology is given by a prebase consisting of certain ultrafilters F and their complements $I = B \setminus F^*$ (ultraideals). The signs $+$, $\cdot$, $'$, Δ will be used to denote, respectively, the operations of taking the sum, intersection, complement, and symmetric difference in a Boolean algebra. Set-theoretic operations of union, intersection, and difference will be denoted, respectively, by $\cup$, $\cap$, $\setminus$; $\emptyset$ denotes the empty set. The zero and unit of a Boolean algebra will be denoted by 0 and 1, respectively.

Definition 1. A system of subsets $\{U_\alpha\}$ of a space X will be called **defining** if the collection of sets $\{U_\alpha\} \cup \{X \setminus U_\alpha\}$ forms a prebase for the topology in X. Each of the subsets U_α and $X \setminus U_\alpha$ will be called **marked** with respect to the defining system $\{U_\alpha\}$.

Lemma 1. Let I, F_1, F be, respectively, an ideal and filters of a Boolean algebra B. If $I \cap F_1 = \emptyset$ and $I \cap F_1 \subseteq F$, then $F_1 \subseteq F$.

Lemma 2. Let $F_1, F_2, \dots, F_n$ be a finite collection of proper filters of a Boolean algebra A, and let F be an ultrafilter distinct from F_i, $i = 1, 2, \dots, n$. Then

$$F \nsubseteq \bigcup_{i=1}^{n} F_i.$$

If each of the filters F_i is an ultrafilter, then, moreover,

$$F \nsubseteq \bigcap_{i=1}^{n} F_i.$$

Lemma 3. The algebraic operations $+$, $\cdot$, $'$, Δ are continuous in any topological algebra B.

Proof. Let $a, b, c \in B$, $a = bc$, and let U be some open set containing the element a. Then for some open ultrafilters F_i and open ultraideals I_j the relations

$$a \in F_1 \cap F_2 \cap \dots \cap F_n \cap I_1 \cap I_2 \cap \dots \cap I_k \subseteq U \tag{1}$$

will hold. For each ideal I_j, in view of its maximality, at least one of the relations $b \in I_j$, $c \in I_j$ must hold. Let $I_{j_1}, I_{j_2}, \dots, I_{j_s}$ be the collection of all those ideals which occur in the notation (1) and do not contain the element b. Then $c \in I = I_{j_1} \cap \dots \cap I_{j_s}$. If $k = s$, then we take as V the set $F = B \cap F_1 \cap \dots \cap F_n$; if $s < k$, then we put $V = F \cap I_0$, where I_0 is the intersection of all ideals from (1) distinct from $I_{j_1}, \dots, I_{j_s}$. Clearly, $b \in V$, $c \in W = I \cap F$, and for any elements $v \in V$ and $w \in W$ the relation $v \cdot w \in U$ is valid. If a' is the complement of the element a in the Boolean algebra B, then the set

$$H = (B \setminus F_1) \cap \dots \cap (B \setminus F_n) \cap (B \setminus I_1) \cap \dots \cap (B \setminus I_k)$$

will contain the element a', and for any element $h \in H$ the relation $h' \in U$ is valid. Since the sets V, W, and H are open, we have thereby proved the continuity of the operations of taking intersection and complement. It remains to note that for any elements $a, b \in B$ the equalities $a + b = (a'b')'$ and $a \Delta b = a'b + b'a$ hold.

Lemma 4. Let $\{F_\alpha\}$ be a defining system of ultrafilters of a topological algebra B, and let F be some ultrafilter in B. The following conditions are equivalent: 1. $F \in \{F_\alpha\}$. 2. F is open-and-closed in B. 3. F is closed in B. 4. F is open in B.

* By an ultrafilter we mean a proper maximal filter.

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are obvious.
$3 \Rightarrow 4$. By Lemma 3, the topological algebra B is a continuous group with respect to the operation Δ. The ideal $I = B \setminus F$, being an open subgroup in B, is closed. Consequently, F is open.

$4 \Rightarrow 1$. We shall show that if $F \notin \{F_\alpha\}$, then F cannot contain any nonempty open set at all. Assuming the contrary, we would have $F \supset F_0 \cap I \neq \emptyset$, where F_0 and I are intersections of a finite number of marked ultrafilters and ultraideals, respectively. But then, by Lemma 1, we have $F_0 \subseteq F$, which contradicts Lemma 2.

Definition 2. Let B be a topological algebra. Denote the set of all its closed ultrafilters by $T(B)$. For each element $b \in B$ define the subset $b_\tau = T(B)$, consisting of all those ultrafilters $F \in T(B)$ that contain the element b. Let $b, b_1, b_2, \dots, b_n$ be arbitrary elements of the topological algebra B. The following relations are easily verified:

$$(b')_\tau = T(B) \setminus b_\tau, \tag{2}$$

$$(b_1 \cdot b_2 \cdots b_n)_\tau = (b_1)_\tau \cap (b_2)_\tau \cap \cdots \cap (b_n)_\tau, \tag{3}$$

$$(b_1 + b_2 + \cdots + b_n)_\tau = (b_1)_\tau \cup (b_2)_\tau \cup \cdots \cup (b_n)_\tau. \tag{4}$$

It is also easy to see that if F_1 and F_2 are distinct ultrafilters from $T(B)$, then there exists an element $c \in B$, $c \in F_1 \setminus F_2$. Then $F_1 \in c_\tau$ and $F_2 \in (c')_\tau$. Therefore, taking as a defining system in the set $T(B)$ the subsets b_τ for all $b \in B$, we turn $T(B)$ into a space. Note that, by relations (2) and (3), the subsets $\{b_\tau\}$, $b \in B$, form a base of open sets of the space $T(B)$.

Definition 3. Let K be a subset of the Boolean algebra B. The set

$$K^* = \{b \in B : \ bc = 0 \text{ for all } c \in K\}$$

will be called the **disjoint complement** of the set K. If $K^{**} = (K^*)^* = K$, then K will be called an algebraic component, or simply a **component**, of the Boolean algebra B. A component K will be called **principal** if there exists an element $c_0 \in K$ such that $K = \{b \in B; \ b \leq c_0\}$. Otherwise the component will be called **nonprincipal**.

Definition 4. A topological algebra B will be called **closed** if, for every nonprincipal component K of it, there exists a closed ultrafilter F in B for which the relation

$$F \cap K = F \cap K^* = \emptyset$$

holds.

Lemma 5. Let B be a closed topological algebra and let U be a clopen subset in $T(B)$. Then there exists an element $b_0 \in B$ for which $(b_0)_\tau = U$.

Proof. Denote by K the set of all those elements b_α of the topological algebra B for which $(b_\alpha)_\tau \subseteq U$, and by P the set of all those elements b_β for which $(b_\beta)_\tau \subseteq V = T(B) \setminus U$; $K = \{b_\alpha\}$, $P = \{b_\beta\}$. Note that

$$\bigcup_\alpha (b_\alpha)_\tau = U, \qquad \bigcup_\beta (b_\beta)_\tau = V.$$

We prove the equality

$$P = K^*. \tag{5}$$

Let $a \in P$, i.e. $a_\tau \subseteq V$. Then $a_\tau \cap U = \emptyset$, hence, for all α,

$$a_\tau \cap (b_\alpha)_\tau = \emptyset,$$

whence $ab_\alpha = 0$; consequently, $a \in K^*$. Conversely, let $a \in K^*$, i.e. $ab_\alpha = 0$ for all α. Then $a_\tau \cap (b_\alpha)_\tau = \emptyset$, whence $a_\tau \cap U = \emptyset$, i.e. $a_\tau \subseteq V$. Consequently, $a \in P$. Interchanging the roles of U and V in this argument, from (5) we obtain $P^* = K$. Consequently,

$$K^{**} = (K^*)^* = P^* = K,$$

i.e. K is a component. Since $U \cup V = T(B)$, every closed ultrafilter F of the topological algebra B belongs either to the set U or to the set V. But this means that either $F \cap K \neq \emptyset$ or $F \cap K^* \neq \emptyset$. Therefore, by the closedness of the topological algebra B, the component K is principal. Let $b_0 \in K$ and, for all α, $b_0 \geq b_\alpha$. Then it is clear that $(b_0)_\tau = U$.

Definition 5. Let X be a space. The Boolean algebra of all its clopen subsets, ordered by inclusion, is denoted···

that is, by $A(X)$. Denote by F_x the ultrafilter of the Boolean algebra $A(X)$ consisting of all open-closed subsets of the space X that contain the point $x \in X$. We introduce a topology in $A(X)$ by taking as a defining system the sets F_x for all $x \in X$. It is clear that this will be a Hausdorff topology. Therefore the algebra $A(X)$ will be a topological algebra.

Lemma 6. *For any space X, the topological algebra $A(X)$ is closed.*

Proof. Suppose, contrary to the assertion of the lemma, that $K = \{a_\alpha\}$ is such a nonprincipal component of the topological algebra $A(X)$ that in $A(X)$ there is no closed ultrafilter F for which $F \cap K = F \cap K^* = \emptyset$. Let $K^* = \{a_\beta\}$. Considering the elements of the topological algebra as subsets of the space X, form the sets $U = \bigcup_\alpha a_\alpha$ and $V = \bigcup_\beta a_\beta$. It is easy to see that $U \cap V = \emptyset$. Let us show that $U \cup V = X$. Indeed, any closed ultrafilter F in $A(X)$, by Lemma 4, is distinguished, $F = F_x$. If $F_x \cap K \neq \emptyset$, then for some $a_\alpha \in K$ we have $a_\alpha \in F_x$, whence $x \in a_\alpha$. If $F \cap K^* \neq \emptyset$, then analogously we obtain $x \in a_\beta$. Thus every point $x \in X$ belongs to the set $U \cup V$, i.e. $U \cup V = X$. Since each of the sets U and V is open, U is open-closed, and hence $U = a$ for some $a \in A(X)$. Since $Ua_\beta = 0$ for all β, we have $U = a \in K^{**} = K$; moreover, $U = a \geq a_\alpha$ for all α, and hence the component K is principal, which contradicts our assumption.

Theorem 1. *Every space X is homeomorphic to the space $T(A(X))$. Every closed topological algebra B is topologically isomorphic to the topological algebra $A(T(B))$.*

Proof. To each point x of the space X assign the closed ultrafilter $\varphi(x) = F_x$ of the topological algebra $A(X)$. Since the elements of the topological algebra $A(X)$ form a base in X, if $x_1 \neq x_2$ then $\varphi(x_1) \neq \varphi(x_2)$. Since every closed ultrafilter F of the topological algebra $A(X)$, by Lemma 4, is distinguished, $F = F_x$ for some $x \in X$, we map this space X onto the whole space $T(A(X))$. It is easy to verify that for any open-closed subset $a \subseteq X$, $a \in A(X)$, the relation $\varphi(a) = a_\tau$ holds. Since the subsets $a \subseteq X$ form a base in the space X, and the subsets $\varphi(a) = a_\tau$ form a base in the space $T(A(X))$, the mapping φ is a homeomorphism.

Now let the topological algebra B be closed. Assign to each element $b \in B$ the set $f(b) = b_\tau$, $b_\tau \subseteq T(B)$. It is clear that if $b_1 \neq b_2$, then $f(b_1) \neq f(b_2)$. Since, in view of the closedness of the topological algebra B, by Lemma 5 every open-closed subset of the space $T(B)$ has the form b_τ, where $b \in B$, f maps B onto the whole topological algebra $A(T(B))$. Moreover, by the relations (2), (3), (4), f is an algebraic isomorphism. Consequently, f carries ultrafilters of the algebra B to ultrafilters of the algebra $A(T(B))$. Let $F = \{b_\alpha\}$ be a closed ultrafilter of the topological algebra B. Then $H = f(F)$ consists of the subsets $(b_\alpha)_\tau$, which are open-closed in the space $T(B)$. From the inclusion $b_\alpha \in F$ it follows that $F \in (b_\alpha)_\tau$ for all α. Hence the ultrafilter H consists of subsets containing the point F of the space $T(B)$, i.e. H is distinguished, and therefore open-closed. Conversely, if $H = \{(b_\alpha)_\tau\}$ is open-closed in the topological algebra $A(T(B))$, then, by Lemma 4, H is a distinguished ultrafilter, i.e. there exists a point F_1 of the space $T(B)$ (F_1 is a closed ultrafilter in B) such that $F_1 \in (b_\alpha)_\tau$ for all α, hence $F_1 \ni b_\alpha$, and therefore $F_1 \supseteq F$, whence $F_1 = F$. Thus $f^{-1}(H) = F$ is an open-closed set in the topological algebra B. This proves the topological character of the isomorphism f.

Corollary 1. *To each space X there corresponds, in view of Lemma 6, the closed topological algebra $A(X)$. To each closed topological-*

Boolean algebra B corresponds to the space $T(B)$. These correspondences are mutually inverse and one-to-one.

Corollary 2. The bicompactness of the space X is equivalent to the fact that in the topological algebra $A(X)$ all ultrafilters are distinguished. In this case any topological algebra is closed, and our theorem turns into the classical Stone theorem.

Theorem 2. The following conditions are equivalent: 1. The space X is discrete. 2. The topology of the topological algebra $A(X)$ coincides with the interval topology. 3. The topology of the topological algebra $A(X)$ is bicompact.

Proof. $1 \Rightarrow 2$. We note that the topology of any topological algebra A is stronger (not weaker) than the interval topology. Let X be discrete. Then any closed ultrafilter in $A(X)$ has the form $[p, 1]$, where p is an atom in $A(X)$, and its complement has the form $[0, p']$, which are, obviously, closed in the interval topology.

$2 \Rightarrow 1$. It suffices to show that any ultrafilter closed in the interval topology is principal. Let $F \subseteq A(X)$ be a closed ultrafilter that is not principal. We have $F = \bigcap_\alpha K_\alpha$, where

$$K_\alpha = [a_1^\alpha, b_1^\alpha] \cup [a_2^\alpha, b_2^\alpha] \cup \ldots \cup [a_{n(\alpha)}^\alpha, b_{n(\alpha)}^\alpha]$$

and for all α the number $n(\alpha)$ is finite. If among the elements $a_1^\alpha, a_2^\alpha, \ldots, a_{n(\alpha)}^\alpha$ there is no zero element, then all the filters $[a_1^\alpha, 1], \ldots, [a_{n(\alpha)}^\alpha, 1]$ are proper filters distinct from F. Therefore, in view of Lemma 2, we have

$$F \nsubseteq [a_1^\alpha, 1] \cup \ldots \cup [a_{n(\alpha)}^\alpha, 1],$$

whence $F \nsubseteq K_\alpha$, which contradicts $F = \bigcap_\alpha K_\alpha \subseteq K_\alpha$. Hence for all α we obtain $0 \in K_\alpha$, whence $0 \in \bigcap_\alpha K_\alpha = F$, which is impossible.

$1 \Rightarrow 3$. If the space X is discrete, then the Boolean algebra $A(X)$ is complete, and by what has been proved the topology on $A(X)$ coincides with the interval topology. Hence ([1]), $A(X)$ is bicompact.

$3 \Rightarrow 1$. We show that any closed ultrafilter of the topological algebra $A(X)$ is principal. Suppose that F is a closed ultrafilter that is not principal; let $\{F_\alpha\}$ be the set of all closed ultrafilters distinct from F. We shall prove that

$$F \subseteq \bigcup_\alpha F_\alpha. \tag{6}$$

Indeed, for any element $a \in F$ one can find elements $b \in A(X)$ and $c \in A(X)$ such that $b \neq a$, $c \neq a$, and $b + c = a$. Then one of the elements b, c belongs to F. Suppose, for example, that $b \in F$. By the Hausdorff property of the topological algebra $A(X)$, there exists a closed ultrafilter F_{α_0} containing the element c. Then $F_{\alpha_0} \neq F$ and $F_{\alpha_0} \ni a$. In view of the arbitrariness of the element a, relation (6) is proved. However, by Lemma 2, from the covering (6) of the closed set F one cannot extract a finite subcover, which contradicts the bicompactness of the topological algebra $A(X)$.

Received
18 II 1970

References

1. G. Birkhoff, *Lattice Theory*, 1952, p. 97, Theorem 15.

Note: Figure translations are in progress. See original paper for figures.