# ON ISOMETRIC OPERATORS ON SUBSPACES OF \(L^p\)

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## Abstract

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*MATHEMATICS*

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# ON ISOMETRIC OPERATORS ON SUBSPACES OF $L^p$

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In this note we generalize and strengthen the results obtained in paper (3), in which some information on the history of the question is also given.

1. Let $(X_1, \sigma_1)$ and $(X_2, \sigma_2)$ be two spaces with positive normalized measures and $0 < p < \infty$. We shall consider complex spaces $L^p(\sigma_j)$, $j = 1, 2$; for brevity of formulation, the number

$$\left( \int_{X_j} |f|^p d\sigma_j \right)^{1/p}$$

will be called the $L^p$-norm of the function $f \in L^p(\sigma_j)$ also in the case $0 < p < 1$.

The basis for all that follows is the following

**Lemma 1.** *Let $p$ be not an even integer and let $f_j \in L^p(\sigma_j)$, $j = 1, 2$. Suppose that for every (complex) $z$*

$$\int_{X_1} |1 + zf_1|^p d\sigma_1 = \int_{X_2} |1 + zf_2|^p d\sigma_2. \tag{1}$$

*Then $|f_1|$ and $|f_2|$ are equidistributed, i.e. for all $\lambda \geq 0$*

$$\sigma_1(\{|f_1| \geq \lambda\}) = \sigma_2(\{|f_2| \geq \lambda\}).$$

**Proof.** Let $\varphi$ and $\psi$ be bounded functions on $[0, 1]$. Put

$$h_1(z) = \int_0^1 \psi(t)\, dt \int_0^{2\pi} |1 + z\varphi(t)e^{i\theta}|^p d\theta,$$

$$h(z) = \int_1^2 ds \int_1^2 h_1(stz)\, dt.$$

Then $h$ is twice continuously differentiable on $(0,\infty)$, $h(z)=h(|z|)$, and for all $r\geq 0$

$$\int_{X_1} h(rf_1)\,d\sigma_1=\int_{X_2} h(rf_2)\,d\sigma_2. \tag{2}$$

Moreover, the functions $\varphi$ and $\psi$ can be chosen so that there exists a number $q$, $0<q<p$, such that

$$\int_0^\infty |h(x)|\,x^{-q-1}dx<\infty.$$

Let $H$ be the Mellin transform of $h$; then $H$ is summable on the line $\operatorname{Re}\zeta=-q$ and, for $x\geq 0$,

$$h(x)=\int_{\operatorname{Re}\zeta=-q} H(\zeta)x^{-\zeta}d\zeta.$$

Hence, for $j=1,2$ and $r\geq 0$ we have

$$\int_{X_j} h(rf_j)\,d\sigma_j=\int_{X_j} h(r|f_j|)\,d\sigma_j=\int_{\operatorname{Re}\zeta=-q} H(\zeta)G_j(\zeta)r^{-\zeta}\,d\zeta, \tag{3}$$

where

$$G_j(\zeta)=\int_{X_j} |f_j|^{-\zeta}\,d\zeta.$$

Taking (2) into account, from (3) we obtain that $G_1=G_2$ on the line $\operatorname{Re}\zeta=-q$. Put $a_j(\lambda)=\sigma_j(\{|f_j|\geq\lambda\})$; then

$$G_j(\zeta)=(-\zeta)\int_0^\infty \lambda^{-\zeta-1}a_j(\lambda)\,d\lambda, \qquad \operatorname{Re}\zeta=-q,$$

and, applying the inverse Mellin transform, we obtain $a_1=a_2$, as was required.

Let now $B$ be a subspace in $L^\infty(\sigma_1)$ containing the constants (not necessarily closed). From Lemma 1 it follows easily

**Lemma 2.** If $p$ is not an even integer and a linear operator $T$ maps $B$ into $L^p(\sigma_2)$ preserving the $L^p$-norms, and moreover $T1=1$, then $T$ preserves the $L^s$-norms for every $s>0$ and the $L^\infty$-norms.

If $p$ is an even integer, then the assertion of Lemma 1 is false. Therefore, in the case of an even integer $p$, generally speaking, the assertion of Lemma 2 is also false unless additional conditions are imposed on the subspace $B$. We shall say that $B$ has property $(\alpha_m)$, $m=1,2,\ldots$, if there exist no more than $m$ subalgebras with 1 in $L^\infty(\sigma_1)$ such that they are contained in $B$, while their (uniformly) closed linear span contains $B$.

**Lemma 3.** If $p$ is an even integer, $p \neq 2$, then the assertion of Lemma 2 is valid under the condition that $B$ has property $(\alpha_{p/2})$.

With the aid of Lemmas 2 and 3 (and also Lemma 2 from [3]) the following theorem on the extension of isometric operators defined on subspaces in $L^p$ is proved.

**Theorem 1.** Let a linear operator $T$ map $B$ into $L^p(\sigma_2)$ with preservation of the $L^p$-norms, and let $p \neq 2$. If $p$ is an even integer, we shall assume that $B$ has property $(\alpha_{p/2})$. Let $F = T1$ and let $\widetilde{B}$ be the smallest symmetric subalgebra in $L^\infty(\sigma_1)$ containing $B$. Then there exists a linear operator $\widetilde{T}$ on $\widetilde{B}$ such that:

1) $\widetilde{T}$ maps $\widetilde{B}$ $L^p$-isometrically into $L^p(\sigma_2)$;
2) $\widetilde{T}$ has the form

$$\widetilde{T}f = F\varphi(f), \qquad f \in \widetilde{B},$$

where $\varphi$ is some $L^\infty$-isometric symmetric homomorphism of the algebra $\widetilde{B}$ into $L^\infty(\sigma_2)$;

3) the restriction of $\widetilde{T}$ to $B$ coincides with $T$.

**Remark.** All the preceding assertions, with the obvious changes in their formulations, are also valid for real $L^p$; we note only that, for even integer $p$, in this case it is enough to require the condition $(\alpha_p)$.

2. Theorem 1 can be applied to find the general form of isometric transformations on certain subspaces in $L^p$. First of all we observe that from this theorem one easily obtains a proof of the well-known Banach–Lamperti theorem (1,2) on the general form of isometric transformations on the whole of $L^p$. Further, if the subspace $B$ is such that the algebra $\widetilde{B}$ is dense in $L^p$, then the problem of describing $L^p$-isometric transformations of $B$ is reduced, in view of Theorem 1, to finding such isometric operators on the whole of $L^p$ with respect to which $B$ is invariant.

Let $D_1$ and $D_2$ be bounded domains in $C^n$. Denote by $L_a^p(D_j)$, $j = 1, 2$, the subspace of $L^p(D_j, d\sigma_j)$ consisting of functions analytic in $D_j$. Here $d\sigma_j$ is the normalized Lebesgue measure in $D_j$.

**Theorem 2.** Suppose that $\operatorname{int}\overline{D}_1 = D_1$ ($\operatorname{int}\overline{D}_1$ is the interior of $\overline{D}_1$) and that $L_a^\infty(D_1)$ is dense in $L_a^p(D_1)$. Let $T$ be an isometric embedding of the space $L_a^p(\underline{D}_1)$ into $L_a^p(D_2)$. Then, if $p \neq 2$, there exists a holomorphic mapping $\tau : D_2 \to \overline{D}_1$ having the following properties:

1) $\tau$ biholomorphically maps the domain

$$D_2 \setminus \{\lambda : \lambda \in D_2,\ \tau'(\lambda) = 0\} \overset{\text{def}}{=} D_2 \setminus J,$$

where $\tau'$ is the Jacobian of $\tau$, onto a domain in $D_1$ whose complement to $D_1$ has measure 0;

2) $(\tau')^{2/p}$ exists as a single-valued holomorphic function in $D_2$;

3) for every $f \in L_a^p(D_1)$

$$Tf = e^{i\gamma}(V_2/V_1)^{1/p}(\tau')^{2/p}f(\tau) \tag{4}$$

on $D_2 \setminus J$ and $Tf = 0$ on $J$, where $\gamma$ is a real constant, and $V_j$ is the volume of $D_j$, $j = 1, 2$.

Conversely, for any holomorphic mapping $\tau : D_2 \to \overline{D}_1$ having properties 1) and 2), and any real $\gamma$, formula (4) defines an isometric embedding $T : L_a^p(D_1) \to L_a^p(D_2)$.

**Remark.** For $n = 1$ and any $p \neq 2$, or for any $n$ but irrational $p$, the formulation of Theorem 2 is simplified. Namely, in this case $J = \emptyset$ (and, consequently, $\tau$ maps $D_2$ into $D_1$). If $n > 1$ and $p$ is rational, then, as simple examples show, $J$ may be nonempty.

Theorem 2 easily implies

**Theorem 3.** Suppose that $\operatorname{int} \overline{D}_j = D_j$ and $L_a^\infty(D_j)$ is dense in $L_a^p(D_j)$, $j = 1, 2$. If the spaces $L_a^p(D_1)$ and $L_a^p(D_2)$ are isometrically isomorphic, then the domains $D_1$ and $D_2$ are biholomorphically equivalent.

3. Let now $D_1$ and $D_2$ be bounded domains in $R^n$, $n > 1$. Denote by $L_h^p(D_j)$, $j = 1, 2$, the subspace in $L^p(D_j, d\sigma_j)$ consisting of functions harmonic in $D_j$.

**Theorem 4.** If $\operatorname{int} \overline{D}_1 = D_1$ and the space $L_h^p(D_1)$ is isometrically isomorphic to some subspace in $L_h^p(D_2)$, $p \neq 2$, then there exists a domain $G_1 \subset D_1$ such that the measure of $D_1 \setminus G_1$ is 0, and, for $p \neq n/(n-2)$, the domains $G_1$ and $D_2$ coincide up to a similarity transformation,* while, if $p = n/(n-2)$, these domains coincide up to similarity transformations and inversion.

If one assumes that $L_h^\infty(D_1)$ is dense in $L_h^p(D_1)$, then in the case when $p$ is not an even integer one can show that every isometric embedding operator of $L_h^p(D_1)$ into $L_h^p(D_2)$ is generated by some similarity transformation (or by a similarity and inversion, for $p = n/(n-2)$). Hence follows

**Corollary.** Let $D$ be a bounded domain in $R^n$ such that $\operatorname{int} \overline{D} = D$ and $L_h^\infty(D)$ is dense in $L_h^p(D)$. If $p$ is not an even integer, then the space $L_h^p(D)$ has no proper subspace isometrically isomorphic to all of $L_h^p(D)$.

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## CITED LITERATURE

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* By a similarity transformation here is meant the product of a homothety and a congruent transformation.

*Note: Figure translations are in progress. See original paper for figures.*

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