# ON THE UNIQUENESS OF THE SOLUTION OF THE CAUCHY PROBLEM FOR A SECOND-ORDER EQUATION WITH VARIABLE COEFFICIENTS

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## Abstract

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*MATHEMATICS*

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# ON THE UNIQUENESS OF THE SOLUTION OF THE CAUCHY PROBLEM FOR A SECOND-ORDER EQUATION WITH VARIABLE COEFFICIENTS

*(Presented by Academician I. N. Vekua, 26 IX 1969)*

This note considers the question of uniqueness of the solution of the Cauchy problem of the form

$$\frac{\partial u(X,t)}{dt} = \frac{1}{a}\Delta u(X,t) + q(X)u(X,t), \qquad u(X,t)\big|_{t=0} = u_0(X),$$

where $a$ is a point of the complex plane, $X \in R_n$ $(n = 1, 2, 3)$, and $\Delta$ is the Laplace operator in $R_n$.

This question, for the case $a > 0$ and a continuous bounded coefficient $q(X)$, $X = (x_1, x_2, \dots, x_n)$, was completely solved in the work of G. N. Zolotarev (1). For the case of unbounded $q(X)$ and $X \in R_1$, the problem was studied in detail by Ya. I. Zhitomirskii (2). He obtained necessary and sufficient conditions for uniqueness of the solution of the Cauchy problem, but the question is considered only for solutions $u(X,t)$ that are defined for all $t > 0$ and grow in $t$ no faster than $e^{ct}$ $(c > 0)$.

The method proposed here, by means of which it is possible to consider the case of many variables, growing $q(X)$, and $t \in [0, T]$, $T < \infty$, consists in reducing the original question to an analogous question for equations with constant coefficients. In the present case this is achieved by using certain properties of solutions of the corresponding hyperbolic equations. For convenience of reading, the method is set out in sufficient detail for the case $X = x \in R_1$.

**Theorem 1.** Let $l(t)$, $t > 0$, be a positive, slowly increasing function, convex upward, for which the integral

$$\int_1^\infty \frac{dt}{t\,l(t)} \tag{1}$$

diverges.

Then the Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{a}\frac{\partial^2 u(x,t)}{\partial x^2} + p(x)\frac{\partial u(x,t)}{\partial x} + q(x)u(x,t), \qquad u(x,t)\big|_{t=0} = u_0(x),$$

$t \in [0,T]$, $-\infty < x < \infty$, can have only one solution $u(x,t)$ belonging, for each $t \in [0,T]$, to the class of functions $f(x)$ satisfying

$$|f(x)| \le C_f \exp\{|x|^2 l(|x|)\}, \tag{2}$$

if $p(x)$ is differentiable and $q(x)$ is continuous, and if these functions satisfy the estimates $(A > 0)$:

$$|q(x)| \le (|x| + A)^2 l^2(|x|), \qquad |p(x)| + |p'(x)| \le (|x| + A)l(|x|).$$

Let us note that the class of functions $f(x)$ satisfying condition (2) ceases to be a uniqueness class for the solution of the given Cauchy problem with constant $p(x)$ and $q(x)$, if $l(t)$ is such that the integral (1) converges (3).

We shall first prove the theorem for the case $p(x) \equiv 0$. Consider the hyperbolic equation

$$\partial^2\varphi(x,y)/\partial y^2 = \partial^2\varphi(x,y)/\partial x^2 + aq(x)\varphi(x,y) \tag{3}$$

and the integral equation associated with it

$$\varphi(x,y) = \varphi_0(x+y) + \frac{a}{2}\int_0^y d\eta \int_{x-(y-\eta)}^{x+(y-\eta)} q(\xi)\varphi(\xi,\eta)\, d\xi, \tag{4}$$

where $\varphi_0(x)$ is a sufficiently smooth finite function.

**Lemma.** Equation (4) has a solution $\varphi(x,y)$, continuous in the whole plane, finite in $x$ for each $y$, and for it, for any $K > 0$, the estimate

$$|\varphi(x,y)| \le M_0 \exp\{(2|a| + K)(|y| + A + r_0)^2 l(2|y| + r_0) - K|x|^2 l(|x|)\}, \tag{5}$$

holds, where $M_0 = \max_x |\varphi_0(x)|$; $r_0$ is a number depending only on $\varphi_0(x)$.

**Proof of the lemma.** For an arbitrary point $(x_0, y_0)$, define the domain $\Omega_0 = \Omega(x_0, y_0)$, consisting of the points

$$(x,y) = (x_0 + \lambda(y_0 - y),\, y),$$

where $\lambda$ varies from $-1$ to 1, and $y$ lies between 0 and $y_0$. Introduce the space $C(\Omega_0)$ of functions $\varphi(x,y)$ continuous on $\Omega_0$ with norm

$$\|\varphi\|_{\Omega_0} = \max_{\Omega_0} |\varphi(x,y)|.$$

If (4) is written in the form $\varphi = \varphi_0 + A\varphi$, where

$$(A\varphi)(x,y) = \frac{a}{2}\int_0^y d\eta \int_{x-(y-\eta)}^{x+(y-\eta)} q(\xi)\varphi(\xi,\eta)\,d\xi,$$

and (4) is solved in $C(\Omega_0)$ by the method of successive approximations ($\varphi_n = \varphi_0 + A\varphi_{n-1}$), then one obtains that the solution exists and

$$|\varphi(x_0,y_0)| \le \sum_{n=0}^{\infty} |(A^n\varphi_0)(x_0,y_0)| \le$$

$$\le \max_{\Omega_0} |\varphi_0(x+y)| \exp\left\{ |y_0| \sqrt{|a| \max_{\Omega_0} |q(\xi)|} \right\}. \tag{6}$$

This estimate is valid for all $x_0, y_0$. In particular, since $\varphi_0(x) \equiv 0$ for $|x| > r_0$, we have $\varphi(x_0,y_0) \equiv 0$ for $|x_0| > |y_0| + r_0$. In the remaining part of the plane, i.e. for $|x_0| \le |y_0| + r_0$, we continue estimate (6), assuming that

$$|q(\xi)| \le \omega(\xi) = (|\xi| + A)^2 l(|\xi|).$$

This gives

$$\max_{\Omega_0} |q(\xi)| \le \omega(|x_0| + |y_0|),$$

whence

$$|\varphi(x_0,y_0)| \le M_0 \exp\{|a|\,|y_0|\sqrt{\omega(2|y_0| + r_0)}\} =$$

$$= M_0 \exp\{|a|\,|y_0|(2|y_0| + r_0 + A) l(2|y_0| + r_0)\} \le$$

$$\le M_0 \exp\{2|a|(|y_0| + A + r_0)^2 l(2|y_0| + r_0)\} \le$$

$$\le M_0 \exp\{(2|a| + K)(|y_0| + A + r_0)^2 l(2|y_0| + r_0) - K|x_0|^2 l(|x_0|)\}.$$

The lemma is proved.

Let us note that the solution $\varphi(x,y)$ under consideration is a solution of equation (3).

We now turn to our Cauchy problem. Let $u(x,t)$ be a solution of the equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{a}\frac{\partial^2 u(x,t)}{\partial x^2} + q(x)u(x,t), \qquad u(x,t)\big|_{t=0} = 0$$

satisfying the condition

$$|u(x,t)| \le C\exp\{|x|^2 l(|x|)\},$$

and let $\varphi(x,y)$–

solution of equation (4). Consider the function

$$F_\varphi(y,t) = (u(x,t),\varphi(x,y)) =$$

$$= \int_{-\infty}^{\infty} u(x,t)\varphi(x,y)\,dx, \qquad t\in[0,T], \quad -\infty < y < \infty.$$

We see that the function has the properties:

a)

$$\frac{\partial}{\partial t}F_\varphi(y,t) = \frac{1}{a}\frac{\partial^2}{\partial y^2}F_\varphi(y,t);$$

b)

$$F_\varphi(y,0) = 0;$$

c)

$$|F_\varphi(y,t)| \le C_1\exp\{C_2(|y|+C_3)^2 l(|y|+C_3)\} \quad (C_1,\ C_2,\ C_3 > 0 \text{ are constants}).$$

Using the results on the uniqueness of the solution of the Cauchy problem for equations with constant coefficients (4), we obtain that $F_\varphi(y,t)\equiv 0$, whence $F_\varphi(0,t) = (u(x,t),\varphi_0(x)) \equiv 0$. This is true for every $\varphi_0(x)\in C_0^\infty$, whence $u(x,t)\equiv 0$, as was required to prove.

If the coefficient $p(x)\ne 0$, then the method of proof remains the same, but in seeking the solution $\varphi(x,y)$ of the hyperbolic equation

$$\frac{\partial^2\varphi(x,y)}{\partial y^2} = \frac{\partial^2\varphi(x,y)}{\partial x^2} - a\frac{\partial}{\partial x}(p(x)\varphi(x,y)) + aq(x)\varphi(x,y)$$

we first make the substitution $\varphi(x,y) = h(x)\varphi_1(x,y)$,

$$h(x) = \exp\left\{\frac{a}{2}\int_0^x p(\xi)\,d\xi\right\},$$

which leads to an equation of the form (3) for $\varphi_1(x, y)$.

This proves the theorem.

In passing to the case $X \in R_n$ $(n = 2, 3)$, we again begin with consideration of the corresponding hyperbolic equations

$$\partial^2\varphi(X, y)/\partial y^2 = \Delta\varphi(X, y) + aq(X)\varphi(X, y).$$

Using the well-known formulas for the solution of hyperbolic equations with a right-hand side, one can compose for these equations ($X \in R_2$ and $X \in R_3$) the corresponding integral equations $\varphi = \varphi_0 + A\varphi$, where as $\varphi_0$ one takes solutions of the problems

$$\partial^2\varphi_0(X, y)/\partial y^2 = \Delta\varphi_0(X, y), \qquad \varphi_0(X, y)\big|_{y=0} = \varphi_0(X) \in C_0^\infty(R_n),$$

$$\partial\varphi_0(X, y)/\partial y\big|_{y=0} = 0.$$

If one additionally assumes that $q(X)$ is continuous and

$$|q(X)| \le (|X| + A)^2 l^2(|X|), \qquad |X| = \left(\sum x_i^2\right)^{1/2},$$

then, by analogy with the case $X \in R_1$, for solutions $\varphi(X, y)$ of the integral equations one can prove the preceding lemma.

As a result, the following assertion can be proved.

**Theorem 2.** Let $l(t)$ satisfy the conditions of Theorem 1, and let $q(X)$, $X \in R_n$ $(n = 2, 3)$, have continuous second partial derivatives with respect to $x_i$, and, for some $A > 0$,

$$|q(X)| \le (|X| + A)^2 l^2(|X|).$$

Then the Cauchy problem

$$\frac{\partial u(X, t)}{\partial t} = \frac{1}{a}\Delta u(X, t) + q(X)u(X, t), \qquad u(X, t)\big|_{t=0} = 0$$

has only the zero solution $u(X, t)$, $t \in [0, T]$, in the class of functions $f(X)$ satisfying the condition

$$|f(X)| \le C_f \exp\{|X|^2 l(|X|)\}.$$

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*