REFLECTION OF SOUND BY AN IMPEDANCE CORRUGATED SURFACE

1970

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Abstract

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UDC 534.26

MATHEMATICAL PHYSICS

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REFLECTION OF SOUND BY AN IMPEDANCE CORRUGATED SURFACE

(Presented by Academician L. M. Brekhovskikh, 27 IV 1970)

Let, in the xy-plane, there be located the trace of a corrugated surface periodic with period $2c$ along the y-axis (in xyz-space, parallel to the z-axis), each period of which consists of a part of a convex curve, having an axis of symmetry parallel to the x-axis, and a part of the straight line $x = 0$ (Fig. 1).

Denote by D the region lying to the right ($x > 0$) of the above-mentioned curve, and let a plane wave be incident from the region D onto its boundary ∂D

$$P_0 = \exp[-ik(\alpha x - \beta y)] \qquad (\alpha^2 + \beta^2 = 1)$$

with wave number k and direction cosines α and β.

Statement of the problem. It is required to find a function $P = P(x, y; k)$, twice continuously differentiable in the region D, continuous on the boundary ∂D, and quadratically integrable in neighborhoods of the corner points of ∂D, regular in k for $|k| < \pi/2c$, which satisfies the following conditions: in the region D, the Helmholtz equation

$$(\Delta + k^2)P = 0; \tag{1}$$

on one part of the boundary ∂D (the set of the above-mentioned convex curves Γ), the Neumann boundary condition

$$\partial P/\partial n\big|_\Gamma = 0 \tag{2}$$

(n is the normal to Γ, directed into D); on the other part of ∂D (the set of segments of the straight line $x = 0$, L), the third boundary condition

$$\partial P/\partial x + ikg(y; k)P\big|_L = 0 \tag{3}$$

Fig. 1

Figure 1: Fig. 1

($g(y;k)$ is a function periodic with period $2c$, continuous in y, regular in k for $|k| < \pi/2c$, satisfying the condition $\operatorname{Re} g \geq 0$); everywhere in the region D, the quasiperiodicity condition

$$P(x, y + 2c) = P(x, y)e^{2ik\beta c}; \tag{4}$$

the Maliuzhinets radiation condition ([1])

$$\sup_D |(P - P_0)e^{-ik\beta y}| < \infty \qquad \text{for } \operatorname{Im} k > |\operatorname{Re} k|. \tag{5}$$

Fig. 1

We note that the problem of diffraction of a plane wave by a rigid, frequently periodic corrugated surface was first posed and solved by G. D. Maliuzhinets. In the present work the method proposed by G. D. Maliuzhinets in solving the above-mentioned problem is applied.

On the basis of the fact that any solution of equation (1) satisfying conditions (4) and (5), as $x \to 0$ and $|k| < \pi/2c$, is representable as–

in the following form ([2]):

$$P = P_0 + ae^{ik(\alpha x + \beta y)} + O(e^{-\sigma x}), \qquad \sigma \geqslant \sigma_0 > 0 \tag{6}$$

(a is a constant), we seek the solution of problem (1)–(5) in the form

$$P = P_0 + \frac{\alpha + iB(\alpha, \beta; k)}{\alpha - iB(\alpha, \beta; k)} e^{ik(\alpha x + \beta y)} + \frac{2\alpha u(x, y; k)}{\alpha - iB(\alpha, \beta; k)} e^{ik\beta y}.$$

Next, substituting P, as defined by the last equality, into conditions (1)–(5), we reformulate the diffraction problem posed above as the following boundary-value problem for the function u:

$$\left(\Delta + 2ik\beta \frac{\partial}{\partial y} + k^2\alpha^2\right) u = 0, \qquad (x, y) \in D; \tag{7}$$

$$\frac{\partial u}{\partial n} + ik\beta \frac{\partial y}{\partial n} u \bigg|_\Gamma = -ik\beta[\cos(k\alpha x) - B\sin(k\alpha x)/\alpha] \frac{\partial y}{\partial n} + k[\alpha \sin(k\alpha x) + B\cos(k\alpha x)] \frac{\partial x}{\partial n} \bigg|_\Gamma; \tag{8}$$

$$\frac{\partial u}{\partial x} + ikgu\bigg|_L = k(B - ig); \tag{9}$$

$$u = O(e^{-\sigma x}) \quad \text{as } x \to \infty; \tag{10}$$

$$u(x, y + 2c) = u(x, y); \tag{11}$$

$$\begin{aligned} 4cB = & -\alpha \int_{\Gamma_0} \sin(2k\alpha x)\frac{\partial x}{\partial n}\,dl - B\int_{\Gamma_0}[\cos(2k\alpha x) - 1]\frac{\partial x}{\partial n}\,dl \\ & - 2\alpha\int_{\Gamma_0} u\sin(k\alpha x)\frac{\partial x}{\partial n}\,dl - 2i\beta\int_{\Gamma_0} u\cos(k\alpha x)\frac{\partial y}{\partial n}\,dl + 2i\int_{L_0}[1 + u(0, y)]g\,dy. \end{aligned} \tag{12}$$

Here Γ_0 and L_0 denote, respectively, the parts of Γ and L for $|y| < c$.
The integral relation (12), connecting B and u, was obtained from Green' s formula for the domain D_0 (D_0 is the part of D for $|y| < c$), applied to the functions P and $\cos(k\alpha x)\exp(-ik\beta y)$.

Using the regularity of the functions u, B, and g with respect to k in a neighborhood of $k = 0$, we expand them in power series in k:

$$B = \sum_{p=0}^{\infty} B^p_{pk}, \qquad u = \sum_{p=0}^{\infty} u^p_{pk}, \qquad g = \sum_{p=0}^{\infty} g^p_{pk}.$$

If the resulting expression is substituted into conditions (7)–(12) and all terms having the same power of k are collected, then, similarly to what was done in ([2]), we obtain a recurrent sequence of boundary-value problems for the Laplace and Poisson equations:

$$\Delta u_p + 2i\beta\frac{\partial u_{p-1}}{\partial y} + \alpha^2 u_{p-2} = 0, \qquad (x, y) \in D; \tag{13}$$

$$\begin{aligned} \frac{\partial u_p}{\partial n} + i\beta\frac{\partial y}{\partial n}u_{p-1}\bigg|_\Gamma = & -i\beta\left[\frac{1-(-1)^p}{2}\frac{(i\alpha x)^{p-1}}{(p-1)!} - \sum_{r=1}^{p-1}\frac{1-(-1)^r}{2i}\frac{(i\alpha x)^r}{\alpha r!}B_{p-r-1}\right]\frac{\partial y}{\partial n} \\ & + \alpha\left[\frac{1+(-1)^p}{2i}\frac{(i\alpha x)^{p-1}}{(p-1)!} + \sum_{r=0}^{p-1}\frac{1+(-1)^r}{2}\frac{(i\alpha x)^r}{\alpha r!}B_{p-r-1}\right]\frac{\partial x}{\partial n}\bigg|_\Gamma; \end{aligned} \tag{14}$$

$$\frac{\partial u_p}{\partial x} + i\sum_{r=0}^{p-1} g_{ru_{p-r-1}}\Big|_L = B_{p-1} - ig_{p-1}; \tag{15}$$

$$u_p = O(e^{-(\pi/c-\varepsilon)x}) \quad \text{as } x \to \infty \qquad (0 < \varepsilon < \pi/c); \tag{16}$$

$$u_p(x, y + 2c) = u_p(x, y); \tag{17}$$

$$4cB_p = -\frac{1-(-1)^p}{2i}\frac{(2i\alpha)^p}{p!}\alpha M_p - \sum_{r=2}^{p}\frac{1+(-1)^r}{2}\frac{(2i\alpha)^r}{r!}B_{p-r}M_r +$$

$$+2i\alpha\sum_{r=1}^{p-1}\frac{1-(-1)^r}{2}\frac{(i\alpha)^r}{r!}\mu^r_{p-r} - 2i\beta\sum_{r=0}^{p-1}\frac{1+(-1)^r}{2}\frac{(i\alpha)^r}{r!}\lambda^r_{p-r} + i\nu_p + i\sum_{r=0}^{p-1}\sigma^r_{p-r}. \tag{18}$$

Here the following notation has been introduced:

$$M_p = \int_{\Gamma_0} x^p \frac{\partial x}{\partial n}\, dl, \qquad \mu^r_p = \int_{\Gamma_0} x^r u_p \frac{\partial x}{\partial n}\, dl, \qquad \lambda^r_p = \int_{\Gamma_0} x^r u_p \frac{\partial y}{\partial n}\, dl,$$

$$\nu_p = \int_{L_0} g_p\, dy, \qquad \sigma^r_p = \int_{L_0} g_r u_p\, dy \tag{19}$$

$$(u_p = B_p = g_p = 0 \quad \text{for } p < 0).$$

In addition, in deriving equality (18) it was taken into account that the function $u_0 \equiv 0$. The latter assertion follows directly from the uniqueness of the solution of the boundary-value problem

$$\Delta u_0 = 0, \quad (x, y) \in D, \qquad \frac{\partial u_0}{\partial n}\Big|_{\partial D} = 0;$$

$$u_0 = O(e^{-\pi x/c}) \quad \text{as } x \to \infty;$$

$$u_0(x, y + 2c) = u_0(x, y).$$

From equality (18), for $p = 0$, we obtain

$$B_0 = \frac{i}{2c}\nu_0 = \frac{i}{2c}\int_{L_0} g_0\,dy. \tag{20}$$

Similarly, from the same equality for $p = 1$, the relation follows

$$B_1 = [i(\nu_1 + \sigma_1^0) - 2\alpha^2 M_1 - 2i\beta\lambda_1^0]/4c \tag{21}$$

(M_1 is the area bounded by the curve Γ_0 and the straight line $x = 0$).

The functionals λ_1^0 and σ_1^0 are determined by the solution of the first boundary-value problem (for the Laplace equation)

$$\Delta u_1 = 0, \quad (x,y) \in D, \qquad \left.\frac{\partial u_1}{\partial n}\right|_\Gamma = -i\beta\frac{\partial y}{\partial n} + B_0\left.\frac{\partial x}{\partial n}\right|_\Gamma,$$

$$\left.\frac{\partial u_1}{\partial x}\right|_L = B_0 - ig_0, \qquad u_1 = O(e^{-\pi x/c}) \quad \text{as } x \to \infty, \tag{22}$$

$$u_1(x, y+2c) = u_1(x,y).$$

Consider a function φ, harmonic in D, periodic with period $2c$, and tending exponentially to zero as $x \to \infty$, whose normal derivative on Γ takes the value

$$\left.\frac{\partial\varphi}{\partial n}\right|_\Gamma = \left.\frac{\partial y}{\partial n}\right|_\Gamma.$$

If Green' s formula for the domain D_0 is applied to the functions u_1 and φ, then we obtain

$$\lambda_1^0 = \frac{1}{2}i\beta\lambda_y, \qquad \lambda_y = -2\int_{\Gamma_0}\varphi\frac{\partial y}{\partial n}\,dl.$$

Here λ_y is the added-mass coefficient of the grating ([2]), obtained from the corrugated surface by adding to the latter its mirror reflection with respect to the y-axis.

Substituting the result obtained above into relation (21),

$$B_1 = [i(\nu_1 + \sigma_1^0) + \beta^2\lambda_y - 2\alpha^2 M_1]/4c. \tag{23}$$

Thus, the field far from the grating in the first approximation in k as $k \to 0$ is determined by the formula

$$P = P_0 + \frac{\alpha + iB_0 + iB_1 k + O(k^2)}{\alpha - iB_0 - iB_1 k + O(k^2)}\, e^{ik(\alpha x + \beta y)};$$

where B_0 and B_1 have been found in the form of expressions (20) and (23) and are entirely and completely computed from the boundary value at L of the solution of the first boundary-value problem and from the coefficient of the attached mass of the corresponding grating.

To find the reflection coefficient of a plane wave from the original corrugated surface in any approximation in k as $k \to 0$, one must use equality (18), whose right-hand side is expressed through the functionals (19), which depend only on the shape of the surface under consideration (the shape coefficients). The above-mentioned shape coefficients are expressed through solutions of the corresponding terms of the recurrent sequence of boundary-value problems for the Poisson equation. Specifying a particular function $g(y; k)$ entering the boundary condition on the corrugated surface, and the shape of this surface, the recurrent sequence mentioned above can conveniently be computed on an electronic computer.

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Received
27 IV 1970

CITED LITERATURE

1. G. D. Malyuzhinets, *Symposium on Wave Diffraction, Abstracts of Reports*, Odessa, 1960, Publishing House of the Academy of Sciences of the USSR, 1960.

2. V. M. Astapenko, G. D. Malyuzhinets, *Acoustical Journal*, vol. 3 (1970).

3. L. I. Sedov, *Plane Problems of Hydrodynamics and Aerodynamics*, Moscow–Leningrad, 1950.

Note: Figure translations are in progress. See original paper for figures.