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EXAMPLES OF BICOMPACTA WITH NONCOINCIDING INDUCTIVE DIMENSIONS

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Abstract

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MATHEMATICS

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EXAMPLES OF BICOMPACTA WITH NON-COINCIDING INDUCTIVE DIMENSIONS

(Presented by Academician P. S. Aleksandrov, 17 X 1969)

In (1) V. Filippov constructed a bicompactum X with $\operatorname{ind} X = 2 < \operatorname{Ind} X = 3$, but the bicompactum X has cardinality 2^{2c}. We constructed a bicompactum T^3 of this note and B. A. Pasynkov constructed bicompacta T^2 and T_1^2 of the note (2) with noncoinciding inductive dimensions and of cardinality c. After becoming acquainted with the bicompactum T^3, applying the device described in item 4 of this note and used by us for the construction of the bicompactum T^3, V. Filippov constructed a bicompactum with the first axiom of countability and with noncoinciding inductive dimensions. Soon after this, both V. Filippov and we constructed simpler bicompacta with the first axiom of countability and with noncoinciding inductive dimensions (see (2, 3) and the bicompacta S^3 of this note).

It is appropriate to note that the constructions of T^3 and S^3 coincide up to replacing the tail χ_1 by the tail χ_2 (see items 5 and 5′).

1. By $R = \{r\}$ and $I = \{i\}$ we shall denote respectively the sets of rational and irrational points of the segment $Q^1 = [0, 1]$. Represent the set R as the disjoint sum of two everywhere dense in Q^1 sets: $R_0 = \{r_0\}$ and $R_1 = \{r_1\}$.
2. Consider the cube $Q^3 = Q^1 \times Q^1 \times Q^1$. For a point $x = (t_0^1, t_0^2, t_0^3) \in Q^3$, let
$$+\tfrac{1}{2}Q_j^3(x) = \{(t^1, t^2, t^3) \in Q^3 :\ t^j \geq t_0^j\}$$
and
$$-\tfrac{1}{2}Q_j^3(x) = \{(t^1, t^2, t^3) \in Q^3 :\ t^j \leq t_0^j\}, \quad j = 1, 2, 3.$$

We shall call the following pairs (x, F) of points $x \in Q^3$ and closed subsets $F \ni x^*$ in Q^3 marked:

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a) $x = (i', i'', i''')$, $F = Q^3$;

b) $x = (k, t', t'')^{**}$, t' and $t'' \in R_{\bar{k}} \cup I$, $F = Q^3$, $k = 0, 1$;

c) $x = (t^1, t^2, t^3)$, $0 < t^1 < 1$, $t^j \in R$, $t^{j'} \in I$ for $j' \neq j$, $F = +\frac{1}{2}Q^3_j(x)$;

d) $x = (k, r_k, t)$, $t \in R_{\bar{k}} \cup I$, $F = +\frac{1}{2}Q^3_2(x)$, $k = 0, 1$;

$x = (k, t, r_k)$, $t \in R_{\bar{k}} \cup I$, $F = +\frac{1}{2}Q^3_3(x)$, $k = 0, 1$;

e) $x = (t^1, t^2, t^3)$, $0 < t^1 < 1$, $t^{j'}$ and $t^{j''} \in R$, $j' \neq j''$,

and $t^j \in I$ for $j \neq j'$ and $j \neq j''$, $F = +\frac{1}{2}Q^3_{j'}(x) \cap +\frac{1}{2}Q^3_{j''}(x)$;

f) $x = (k, r_k, r'_k)$, $F = +\frac{1}{2}Q^3_2(x) \cap +\frac{1}{2}Q^3_3(x)$, $k = 0, 1$;

g) $x = (r^1, r^2, r^3)$, $0 < r^1 < 1$, $F = \bigcap_{j=1}^{3} +\frac{1}{2}Q^3_j(x)$.

(1)

* To shorten the notation we shall adopt the following convention. If a pair is marked
1) $(x, +\frac{1}{2}Q^3_j(x))$, or 2) $(x, +\frac{1}{2}Q^3_{j'}(x) \cap +\frac{1}{2}Q^3_{j''}(x))$, or 3) $(x, +\frac{1}{2}Q^3_{j'}(x) \cup +\frac{1}{2}Q^3_{j''}(x))$, or 4) $(x, \bigcap_{j=1}^{3} +\frac{1}{2}Q^3_j(x))$, or 5) $(x, +\frac{1}{2}Q^3_{j'''}(x) \cup +\frac{1}{2}Q^3_{j'}(x) \cap +\frac{1}{2}Q^3_{j''}(x))$, then in each of the cases 1)–5) all pairs obtained from the indicated pair by replacing, in its second element, some (or all) plus signs by minus signs are also regarded as marked. The indices j, j', j'', j''' may take the values $1, 2, 3$.

** Everywhere k is equal either to 0 or to 1 and, if $k = 0$, then $\bar{k} = 1$, while if $k = 1$, then $\bar{k} = 0$.

(2) h) $x = (t^1, t^2, t^3)$, $0 < t^1 < 1$, $t^{j'} \in R_0$, $t^{j''} \in R_1$, $j' \neq j''$, $t^j \in I$ for $j \neq j'$ and $j \neq j''$,

$$F = +^{1/2}Q^3_{j'}(x) \cup +^{1/2}Q^3_{j''}(x);$$

i) $x = (t^1, t^2, t^3)$, $0 < t^i < 1$, $t^{j'}$ and $t^{j''} \in R_k$, $j' \neq j''$, $t^{j'''} \in R_{\bar{k}}$, $j''' \neq j'$ and $j''' \neq j''$,

$$F = +^{1/2}Q^3_{j'''}(x) \cup (+^{1/2}Q^3_{j'}(x) \cap +^{1/2}Q^3_{j''}(x)).$$

Obviously, the cardinality of the set B of all marked pairs is $\mathfrak{c}$. The set of the pairs marked in items a)–g) will be denoted by B_1, and the set of the pairs marked in items h) and i), by B_2.

3. Let a bicompactum P_0 be contained in a bicompactum P_1 and have a continuous mapping f onto a bicompactum P. As the elements of the decomposition τ of the bicompactum P_1 we take the points of the set $P_1 \setminus P_0$ and the inverse images of points of the bicompactum P under the mapping f. We shall say that the bicompactum P_1^*, which is the

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decomposition space τ, is obtained from the bicompactum P_1 by gluing the bicompactum P_0 to the bicompactum P by means of the mapping f.

4. Let a bicompactum X and a system of pairs (x_θ, F_θ), $\theta \in \Theta$, of points $x_\theta \in X$ and closed subsets $F_\theta \ni x_\theta$ in X be given. Let there also be given a bicompactum χ and in it a disjoint system of open sets C_θ, $\theta \in \Theta$, each of which decomposes into the disjoint sum of bicompacta C_α, $\alpha \in \mathfrak{A}_\theta$, open-and-closed in χ. Suppose that for each α a mapping $f_\alpha : C_\alpha \to \Phi_\alpha$ is defined. In the bicompactum

$$X \times \chi \setminus \bigcup_{\theta\in\Theta} \bigcup_{\alpha\in\mathfrak{A}_\theta} (X \setminus F_\theta \times C_\alpha)$$

for each $\alpha \in \mathfrak{A}_\theta$ we glue the bicompactum $x_\theta \times C_\alpha$ to the bicompactum Φ_α by means of the mapping f_α, $\theta \in \Theta$. The resulting bicompactum will be denoted by

$$D(X, \chi, \{x_\theta, F_\theta, C_\alpha, \Phi_\alpha, f_\alpha;\ \alpha \in \mathfrak{A}_\theta\},\ \theta \in \Theta).$$

5. By the tail χ_1 we shall mean the ordered zero-dimensional bicompactum obtained as follows. Take the set

$$\chi_1 = \bigcup_{\alpha\leq\omega(\mathfrak{c})} \alpha \ \cup \bigcup_{\alpha<\omega(\mathfrak{c})} C_\alpha,$$

where $\omega(\mathfrak{c})$ denotes the first ordinal of cardinality $\mathfrak{c}$, α are ordinals, and C_α are Cantor perfect sets.

The sets C_α are taken in their natural topology and are regarded as open-and-closed in χ_1; as a basic neighborhood of a nonlimit number α we take the number itself, and as basic neighborhoods of a limit number α' we take sets of the form

$$\bigcup_{\alpha''<\alpha\leq\alpha'} \alpha \ \cup \bigcup_{\alpha''<\alpha<\alpha'} C_\alpha.$$

Represent the set

$$\mathfrak{A} = \{\alpha < \omega(\mathfrak{c})\}$$

as a disjoint sum of sets $\mathfrak{A}_\theta$, $\theta \in \Theta$, of cardinality $\mathfrak{c}$, so that the cardinality of Θ is also equal to $\mathfrak{c}$.

Between the sets Θ and B (see item 2) we establish a one-to-one correspondence.

6. We now construct a bicompactum T^3 with

$$\dim T^3 = \operatorname{ind} T^3 = 3 < \operatorname{Ind} T^3 = 4.$$

Consider the product $Q^3 \times \chi_1$. Fix some mapping f of the Cantor set C onto Q^3 and some mapping g of the Cantor set C onto the square Q^2. Then

$$T^3 = D(Q^3, \chi_1, \{x_\theta, F_\theta, C_\alpha, \Phi_\alpha, f_\alpha;\ \alpha \in \mathfrak{A}_\theta\},\ \theta \in \Theta),$$

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where (x_θ, F_θ) is the element of the set B (i.e. the marked pair) corresponding to the element θ of the set Θ; moreover, if $(x_\theta, F_\theta) \in B_1$, then $\Phi_\alpha = Q^3$ and $f_\alpha = f$, while if $(x_\theta, F_\theta) \in B_2$, then $\Phi_\alpha = Q^2$ and $f_\alpha = g$.

The bicompactum T^3 has cardinality $\mathfrak{c}$ and decomposes into the sum of the cube $Q^3 \times \omega(\mathfrak{c})$ and an additional set which is locally metrizable if $\mathfrak{c} = \aleph_1$. Let us also note that the set

$$T^3 \setminus Q^3 \times \{\alpha \leq \omega(\mathfrak{c})\}$$

is metrizable.

5′. By the tail χ_2 we shall mean the ordered zero-dimensional bicompactum with the first axiom of countability, obtained as follows.

Consider the lexicographically ordered product

$$U = Q^1 \times D$$

of the interval $Q^1 = [0,1] = \{u\}$ and the pair of isolated points

$$D = \{0,1\}.$$

This bicompactum was constructed in (4) and is called "two arrows." Let $u_\ell = (u,0)$, $u_p = (u,1)$. Put in correspondence with each number u the lexicographically ordered product $V_u = Q^1_u \times D$ of the interval $Q^1_u = [0,1] = \{v_u\}$ and the pair of isolated points $D = \{0,1\}$. Let $v_{u\ell} = (v_u, 0)$ and $v_{ur} = (v_u, 1)$. Finally, to each number $v_u \in Q^1_u$, $u \in [0,1]$, we assign the Cantor set C_{v_u}. Then

$$\chi_2 = U \cup \bigcup_{0 \leq u \leq 1} V_u \cup \bigcup_{\substack{0 \leq u \leq 1 \\ 0 \leq v_u \leq 1}} C_{v_u}.$$

The elements of the sets C_{v_u}, V_u, and U have already been ordered. We put $v_{u\ell} < c_{v_u} < v_{ur}$ for any point $c_{v_u} \in C_{v_u}$, and $u_\ell < v_u < u_r$ for any point $v_u \in V_u$. The order relation for the other pairs of points is defined by transitivity. The order (interval) topology gives the required bicompactum χ_2. Represent the interval $Q^1_u = [0,1] = \{v_u\}$ as a disjoint sum of everywhere dense sets $\mathfrak{A}_{\theta u} = \{a = v_u\}$, $\theta \in \Theta$, such that the cardinality of Θ is $\mathfrak{c}$, $0 \leq u \leq 1$. Let $\mathfrak{A}_\theta = \bigcup_{0 \leq u \leq 1} \mathfrak{A}_{\theta u}$, $\theta \in \Theta$. Between the sets Θ and B establish a one-to-one correspondence.

6′. We now construct a bicompactum S^3 with the first axiom of countability and with

$$\dim S^3 = \operatorname{ind} S^3 = 3 < \operatorname{Ind} S^3 = 4.$$

Consider the product $Q^3 \times \chi_2$. Then

$$S^3 = D(Q^3, \chi_2, \{x_\theta, F_\theta, C_\alpha, \Phi_\alpha, f_\alpha, \alpha \in \mathfrak{A}_\theta\},\ \theta \in \Theta),$$

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where (x_θ, F_θ) is the element of the set B (i.e. the marked pair) corresponding to the element θ of the set Θ; moreover, if $(x_\theta, F_\theta) \in B_1$, then $\Phi_\alpha = Q^3$ and $f_\alpha = f$ (see item 6), while if $(x_\theta, F_\theta) \in B_2$, then $\Phi_\alpha = Q^2$ and $f_\alpha = g$.

The bicompactum S^3 decomposes into the sum: a) of the product $Q^3 \times (U \cup \bigcup_u V_u)$ of the cube and a zero-dimensional ordered bicompactum with the first axiom of countability, and b) of a metrizable complement to this product*.

7. The method of constructing the bicompacta T^3 and S^3 is of a general nature.

Theorem. For any $n \geq 3$ there exist bicompacta T^n and S^n of cardinality $\mathfrak{c}$ and dimension

$$\dim T^n = \operatorname{ind} T^n = \dim S^n = \operatorname{ind} S^n = n < \operatorname{Ind} T^n = \operatorname{Ind} S^n = n + 1,$$

1) The bicompactum T^n decomposes into the sum: a) of the product of the n-dimensional cube Q^n and the ordinal numbers $\leq \omega(\mathfrak{c})$, b) of a metrizable complement to this product*. If $\mathfrak{c} = \aleph_1$, then the complement to the cube $Q^n \times \omega(\mathfrak{c})$ is locally metrizable.

2) The bicompactum S^n has the first axiom of countability and decomposes into the sum: a) of the product of the n-dimensional cube Q^n and a zero-dimensional ordered bicompactum with the first axiom of countability, b) of a metrizable complement to this product*.

Corollary. There exist locally bicompact metrizable spaces of weight and cardinality $\mathfrak{c}$ having bicompact extensions (even with the first axiom of countability) with different inductive dimensions.

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[3] V. Filippov, DAN, 1970.
[4] P. S. Aleksandrov, P. S. Uryson, in the book: P. S. Uryson, *Works on Topology and Other Fields of Mathematics*, 2, Moscow-Leningrad, 1951, p. 848.

* Which is a discrete sum of compacta.

Note: Figure translations are in progress. See original paper for figures.

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