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Abstract

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MATHEMATICS

Yu. N. SUBBOTIN

APPROXIMATION OF FUNCTIONS OF THE CLASS $W^k H^p_\omega$ BY SPLINES OF ORDER m

(Presented by Academician I. M. Vinogradov on 18 V 1970)

Let $f(x) \in W^k_p$, i.e., it has an absolutely continuous $(k-1)$-st derivative and

$$\|f^{(k)}(x)\|_{L_p(0,1)} = \left\{ \int_0^1 |f^{(k)}(x)|^p \, dx \right\}^{1/p} < \infty \qquad (1 \le p \le \infty). \tag{1}$$

In this paper we consider questions of approximation of such functions on $[0,1]$ by splines $S_{m,n}(x)$ of order m $(m \ge k)$ with a number of knots $\{x_s\}$ not exceeding n, in the metric L_q. The case $m = k-1$ was considered in (^{1}). The spline $S_{m,n}(x)$ is a piecewise-polynomial function, glued from polynomials of order not higher than m in such a way that the derivative $S^{(m-1)}_{m,n}(x)$ is continuous on $[0,1]$, while $S^{(m)}_{m,n}(x)$ has discontinuities only at the knots $\{x_s\}$. The cases of fixed and nonfixed placement of knots are considered. For $p = q = \infty$, for fixed knots $\{x_s\}$ and odd splines, there is a rather extensive literature devoted to these questions (for a bibliography see, for example, (^{2})); there, under a certain relation between k and m, the case $p = 2,\ q = \infty$ is also considered. We also note an interesting result of V. M. Tikhomirov (^{3}), pertaining to the case $p = q = \infty$. In the present paper we also consider approximations of functions $f(x)$, defined on the whole real axis and belonging to the class $W^k H^p_\omega$, by interpolating splines $S_m(x,h)$ (^{4,5}) of order m, which interpolate the function $f(x)$ on the uniform grid $\{sh\}$ $(s = 0, \pm 1, \pm 2, \ldots)$. In what follows,

$$\omega_p(f^{(k)}, h) = \sup_{|t| \le h} \left\{ \int_0^1 |f^{(k)}(x+t) - f^{(k)}(x)|^p \, dx \right\}^{1/p},$$

where in the nonperiodic case the derivative $f^{(k)}(x)$ outside the interval $[0,1]$ may be extended, for example, by zero.

Theorem 1. If the function $f(x) \in W^k_p$, then for any $p, q \ge 1$, including the case $k = 1,\ 1 \le p < 2q(q+1)^{-1},\ p < q$, the inequality

$$\inf \|f - S_{k,n}\|_{L_p(0,1)} \le Cn^{-k}\omega_p(f^{(k)}, n^{-1}), \tag{2}$$

holds, where the infimum is taken over all splines $S_{k,n}(x)$ of order k with a number of knots not exceeding n, and the constant C depends only on k.

Theorem 2. If the function $f(x) \in W_p^k$ and $m \ge k$, then

$$\inf_{S_{m,n}} \|f - S_{m,n}\|_{L_p(0,1)} \le C(k,m)n^{-k}\|f^{(k)}\|_{L_p(0,1)} \qquad (p, q \ge 1), \tag{3}$$

where $S_{m,n}(x)$ is a spline of order m with a number of knots not exceeding n.

For $m = k - 1$ inequality (3) was proved in (^{1}).

Theorem 3. There exist splines $S_{m,n}(x)$ and $S_{k,n}(x)$ such that $(m \ge k)$

$$\|f^{(i)} - S_{m,n}^{(i)}\|_{L_p(0,1)} \le C(k,m)n^{-k+i}\|f^{(k)}\|_{L_p(0,1)} \quad (p, q \ge 1,\ 0 \le i \le k), \tag{4}$$

$$\|f^{(i)} - S_{k,n}^{(i)}\|_{L_q(0,1)} \le C(k)n^{-k+i}\omega_p(f^{(k)}, n^{-1}) \quad (p, q \ge 1,\ 0 \le i < k,\ k > 1). \tag{5}$$

The example of the function $f_n(x)$,

$$f_n^{(k)}(x) = (x - x_s)^{m+1-k}n^{m+1-k}, \quad x_s \le x \le x_{s+1},$$

$x_s = s/n$ $(s = 0, 1, \dots, n-1)$, shows that estimate (3) cannot be improved.

Let a sequence of meshes be given

$$\Delta_s: \ 0 = x_0^{(s)} < x_1^{(s)} < \dots < x_{n_s}^{(s)} = 1, \quad n_s = sm \quad (s = 1, 2, \dots). \tag{6}$$

Theorem 4. If the quantities $(t_i = x_{im})$

$$R_s = \max_{0 \le i \le s-1} \max_{t_i \le t \le t_{i+1}} \sum_{r=0}^{m-1} \frac{\psi_i(t)}{\psi_i'(x_{im+r})(t - x_{im+r})} \le \bar{\beta} < \infty, \tag{7}$$

$$\psi_i(t) = (t - x_{im})(t - x_{im+1}) \cdots (t - x_{i(m+1)}),$$

then there exists a spline $S_{m,n_s}(x)$ with knots (6) such that

$$\|f^{(i)} - S_{m,n_s}^{(i)}\|_{L_q(0,1)} \le C_1\|\Delta_s\|^{\gamma - i}\|f^{(k)}(x)\|_{L_p(0,1)}, \tag{8}$$

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where

$$\|\Delta_s\| = \max_{0 \le i \le n_s - 1} |x_{i+1}^{(s)} - x_i^{(s)}|, \quad \gamma = k \ (q \le p) \quad \text{and} \quad \gamma = k + q^{-1} - p^{-1} \ (q > p).$$

If $m = k$, then $\|f^{(k)}\|_{L_p(0,1)}$ may be replaced by $\omega_p(f^{(k)}, \|\Delta_s\|)$.

If the spline $S_m(x, h)$ interpolates the function $f(x) \in W^k L_p(-\infty, \infty)$ at the knots sh $(s = 0, \pm 1, \pm 2, \dots)$, then the following holds.

Theorem 5. For $q \ge p$ and $m > k \ge 1$ the inequality

$$\|f^{(i)}(x) - S_m^{(i)}(x, h)\|_{L_p(-\infty,\infty)} \le C h^{k+1/q-1/p-i} \omega_p(f^{(k)}, h) \quad (0 \le i \le k-1), \tag{9}$$

holds, where C depends only on k and m, and

$$\omega_p(f^{(k)}, h) = \sup_{|t| \le h} \|f^{(k)}(x+t) - f^{(k)}(x)\|_{L_p(-\infty,\infty)}. \tag{10}$$

In the proof of Theorems 1 and 2, a certain extremal set of knots is constructed (for each function its own)

$$0 = x_0 < x_1 < \dots < x_s = 1,$$

and then, on each interval $[x_i, x_{i+1}]$, the spline $S_{m,m,i}(x)$ is determined from the conditions

$$f^{(\alpha)}(x_r) = S_{m,m,i}^{(\alpha)}(x_r), \quad \alpha = 0, 1, \dots, k-1; \tag{11}$$

$$S_{m,m,i}^{(\alpha)}(x_r) = 0, \quad \alpha = k, \dots, m-1 \ (m > k); \ r = i, i+1.$$

Here the intermediate knots of the spline $S_{m,m,i}(x)$ are chosen in the form

$$x_i^{(l)} = x_i + \lambda_l (x_{i+1} - x_i),$$

where the numbers λ_l do not depend on i and satisfy the inequalities

$$0 = \lambda_0 < \lambda_1 < \dots < \lambda_m = 1.$$

The proof of Theorem 1 reduces to computing the quantity

$$\beta = \inf_{\{x_i\}} \max_{0 \le i \le s-1} \beta_i, \tag{12}$$

$$\beta_i = \Delta x_i^{kp-2+p/q} \int_0^{\Delta x_i} dt \int_0^{\Delta x_i} |f^{(k)}(t+u+x_i) - f^{(k)}(u+x_i)|^p \, du, \tag{13}$$

where $\Delta x_i = x_{i+1} - x_i$. For $kp - 2 + p/q \geq 0$, just as in (1), it is also proved that the infimum is attained when all β_i are equal. The proof of Theorem 2 proceeds analogously, only in the present case

$$\beta_i = \Delta x_i^{kp-1+p/q} \int_0^{\Delta x_i} \left|f^{(k)}(u+x_i)\right|^p \, du. \tag{14}$$

Comparison of Theorems 1 and 2 with Theorem 4 shows that for $q > p$ a special choice of knots ensures a better order of convergence.

In the proof of Theorem 4 the representation for $S_m^{(m)}(x,h)$ found in (5) is used. For $p = q = \infty$ this theorem was proved in (6).

Sverdlovsk Branch of the V. A. Steklov Mathematical Institute Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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