INVARIANCE GROUPS AND DIFFERENTIATION IN THE THEORY OF MATRIX SPACE

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INVARIANCE GROUPS AND DIFFERENTIATION IN THE THEORY OF MATRIX SPACE

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1. A new covariant approach to the theory of the spinor field, set forth in paper (^{1}), requires the introduction of the concept of matrix space. In the present article the basic properties of matrix space connected with the existence of invariance groups are discussed. The rules for differentiating matrix tensors are established.

Let 4 matrices 4×4, $A^\alpha(x)$, be given at each point of a Riemannian space of general relativity. If under coordinate transformations they transform as the components of a 4-vector, then $A^\alpha(x)$ will be called a matrix vector. A matrix tensor with an arbitrary number of contravariant and covariant indices is defined in an analogous way. The admissible algebraic operations for matrix tensors include matrix multiplication, addition, the operation of taking a trace, contraction over tensor indices, complex conjugation, transposition of matrices, and Hermitian conjugation. Matrix space is defined by the fact that in Riemannian space there is given a matrix vector $\gamma^\alpha(x)$ (4 Dirac matrices) satisfying the relation:

$$[\gamma^\alpha(x), \gamma^\beta(x)]_+ = 2g^{\alpha\beta}(x) \cdot E; \tag{1}$$

here $g^{\alpha\beta}(x)$ is the metric tensor.

An arbitrary algebraic function of γ^α using all the algebraic operations listed above is also, by our definition, a matrix tensor. In what follows we shall confine our consideration to only such matrix tensors, including covariant differentiation among the admissible operations.

Matrix tensors composed of γ^α are, for example,

$$S^{\alpha\beta} = {}^1/_2(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha), \qquad \gamma_5 = {}^1/_{24}E_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu. \tag{2}$$

Along with (2), $\gamma^{\alpha *}$, $\gamma^{\alpha T}$, $\gamma^{\alpha +}$, and any algebraic functions of γ^α, for example, $F^\alpha = \gamma^\nu\gamma^\alpha\gamma^{\nu *} + \gamma^{\alpha T}\gamma^\nu\gamma^\mu T\gamma^{\nu *}\gamma_\mu^T$.

2. It is known (4) that, for a given $g^{\alpha\beta}(x)$, equality (1) determines the set of Dirac matrices up to an arbitrary unimodular $S(x)$-transformation,

$$\gamma^{\alpha}(x) \to \gamma {a'}^{\alpha}(x) = S(x)\gamma^{\alpha}(x)S^{-1}(x), \qquad \det[S(x)] = 1. \tag{3}$$

Suppose that in matrix space certain algebraic relations between matrix tensors are given. Under transformations (3), algebraic relations that include, along with γ^{α}, $\gamma^{\alpha *}$, $\gamma^{\alpha T}$, and $\gamma^{\alpha +}$, generally speaking, are violated. We shall find, among the 30-parameter S-transformations, those particular transformations $T(x)$,

$$\gamma^{\alpha}(x) \to \gamma {a'}^{\alpha}(x) = T(x)\gamma^{\alpha}(x)T^{-1}(x), \qquad \det[T(x)] = 1, \tag{4}$$

which do not violate any algebraic relations between matrix tensors and therefore may be called transformations of the representation of matrix tensors, in no way connected with the choice of the coordinate system. The matrix vectors $\gamma^{\alpha *}$, $\gamma^{\alpha T}$, $\gamma^{\alpha +}$ under transformation (3) pre-

are formed as follows:

$$\gamma^{\alpha *} \to \gamma {a'}^{\alpha *} = S^{*}\gamma^{\alpha *}S^{*-1}, \qquad \gamma^{\alpha T} \to \gamma {a'}^{\alpha T} = S^{-1T}\gamma^{\alpha T}S^{T},$$
$$\gamma^{\alpha +} \to \gamma {a'}^{\alpha +} = S^{-1+}\gamma^{\alpha +}S^{+}, \tag{5}$$

and the form of the algebraic relations will be preserved only under such S-transformations for which

$$S^{*}\gamma^{*}_{\alpha}S^{*-1} = S\gamma^{*}_{\alpha}S^{-1}, \qquad S^{-1T}\gamma^{T}_{\alpha}S^{T} = S\gamma^{T}_{\alpha}S^{-1}, \qquad S^{-1+}\gamma^{+}_{\alpha}S^{+} = S\gamma^{+}_{\alpha}S^{-1} \tag{6}$$

The matrix S must satisfy the relations $S^{*} = \lambda \cdot S$, $S^{T} = \mu \cdot S^{-1}$, $S^{+} = \nu \cdot S^{-1}$, where λ, μ, ν are numbers, with $\lambda\mu = \nu$ and $\lambda^{*}\nu = \mu$. Taking into account unimodularity (3), and also bearing in mind that the S-matrix is determined up to multiplication by $\pm 1, \pm i$, we obtain the desired T-matrices:

$$T^{*} = T, \qquad T^{T} = T^{-1}, \qquad T^{*} = -iT, \qquad T^{T} = iT^{-1}. \tag{7}$$

Matrices with the properties (7) constitute a 6-parameter T-group, isomorphic to the group of proper and improper rotations of four-dimensional Euclidean space. Under a T-transformation, obviously, every algebraic matrix tensor $A(x)$ is transformed according to the law

$$A(x) \to A'(x) = T(x)A(x)T^{-1}(x). \tag{8}$$

It is easy to verify that a simultaneous permutation of the rows and columns in all matrix tensors is a T-transformation. A general T-transformation may also be interpreted as a transition to another method of denoting matrix elements. Since no algebraic operation allows one to distinguish any one of the admissible methods, T-invariance is the independence of the factual relation between the matrix elements of any matrix relation from the choice of an admissible method of denoting matrix elements.

3. In finding the general form of the covariant derivative $\nabla_\lambda A^{\alpha\beta\cdots}_{\mu\cdots}$, we shall, as usual, assume that the conditions of associativity and distributivity are fulfilled, $\nabla(A+B)=\nabla A+\nabla B$, $\nabla(AB)=(\nabla A)B+A(\nabla B)$ (here the tensor signs are omitted for simplicity), and we shall also proceed from the fact that $\nabla_\nu A$ coincides with the covariant derivative in the ordinary tensor sense, $A_{;\nu}$, if A is a numerical tensor.

Covariantly differentiating (1), we obtain

$$[\nabla_\nu\gamma^\alpha,\gamma^\beta]_+ + [\gamma^\alpha,\nabla_\nu\gamma^\beta]_+ = 0. \tag{9}$$

The general form of the solution of (9) is

$$\nabla_\nu\gamma^\alpha = [k_\nu,\gamma^\alpha]_-, \tag{10}$$

where k_ν is a set of four matrices transforming as the components of a 4-vector. In an analogous way we establish that

$$\gamma^\alpha{}_{;\nu} = -[\Phi_\nu,\gamma^\alpha]_-. \tag{11}$$

Relations (10), (11) lead to

$$\nabla_\nu\gamma^\alpha = \gamma^\alpha{}_{;\nu} + [\Gamma_\nu,\gamma^\alpha]_-, \qquad \Gamma_\nu = k_\nu + \Phi_\nu. \tag{12}$$

Writing, for an arbitrary matrix tensor A, the expansion

$$A = \frac{1}{4}E\cdot\mathrm{Sp}\,A + \frac{1}{4}\gamma^\nu\cdot\mathrm{Sp}\,A\gamma_\nu + \frac{1}{8}S^{\mu\nu}\cdot\mathrm{Sp}\,AS_{\mu\nu} - \frac{1}{4}\gamma_5\gamma^\nu\cdot\mathrm{Sp}\,A\gamma_5\gamma_\nu - \frac{1}{4}\gamma_5\cdot\mathrm{Sp}\,A\gamma_5 \tag{13}$$

and covariantly differentiating (13) with the use of (12), we obtain

$$\nabla_\nu A = A_{;\nu} + [\Gamma_\nu, A]_-, \tag{14}$$

whence it follows that, in order to find the covariant derivative of an arbitrary matrix tensor, it is necessary to know Γ_ν.

Let $A(x)$ be a field of a matrix vector. By definition

$$dx^\nu(\nabla_\nu A^\alpha) = {A'}^\alpha - A^\alpha(x), \tag{15}$$

where ${A'}^\alpha$ is the vector $A^\alpha(x+dx)$, parallel transported to the point x. The change $dx^\nu(\nabla_\nu A^\alpha)$ includes, along with $dx^\nu({A^\alpha}_{;\nu})$, the change $dx^\nu(\Gamma_\nu, A^\alpha)_-$.

connected with the difference of the matrix properties at the points x and $x+dx$, is evident from (14) and (15).

Let there exist certain covariant relations between the numerical fields. Then, as is known, under parallel transport tensors change synchronously, i.e., the relations between the fields do not change their form. In the case of parallel transport of matrix tensors we shall likewise require that all covariant algebraic relations between them not change their form.

Suppose that we parallel-transport from $x+dx$ to x a material matrix tensor A, $A^*(x+dx) = A(x+dx)$. Then at the point x it must be $(A')^* = A'$, whence, taking (14), (15) into account, it follows that for any material matrix tensor A one has $\nabla_\nu(A^*) = (\nabla_\nu A)^*$, which can be the case only if

$$\Gamma_\nu^* = \Gamma_\nu. \tag{16}$$

Similarly it is proved that

$$\Gamma_\nu^T = -\Gamma_\nu, \qquad \Gamma_\nu^+ = -\Gamma_\nu. \tag{17}$$

Thus, from the properties of parallel transport we have established that Γ_ν is a set of real anti-Hermitian matrices.

4. From definition (15) it follows that under a T-transformation, ∇A behaves as an algebraic matrix tensor, i.e. $(\nabla A) \to T(\nabla A)T^{-1}$; on the other hand, we have $(\nabla A) \to \nabla'(TAT^{-1})$, where ∇' means that the derivative is taken in the $(T\gamma T^{-1})$-field. Comparison of these two formulas gives

$$\nabla'(TAT^{-1}) = T(\nabla A)T^{-1}. \tag{18}$$

The vector Γ_ν depends linearly on the vector Φ_ν and vanishes together with $\Phi_\nu = 0$. This follows from the definitions of Γ_ν, Φ_ν. While Γ_ν is anti-Hermitian and material, no such restrictions are imposed on Φ_ν. To find Γ_ν, we must form from Φ_ν such a linear combination which is anti-Hermitian and material and which, under T-transformations, behaves, as follows from (18), in the following way:

$$\Gamma_\nu' = T\Gamma_\nu T^{-1} + T(\partial T^{-1}/\partial x^\nu). \tag{19}$$

From (11) it follows that under arbitrary S-transformations

$$\Phi'_\nu = S\Phi_\nu S^{-1} + S(\partial S^{-1}/\partial x^\nu); \tag{20}$$

in particular, for $S = T$,

$$\Phi'_\nu = T\Phi_\nu T^{-1} + T(\partial T^{-1}/\partial x^\nu). \tag{21}$$

Using the properties of T-matrices (7), from (21) we obtain

$$^1/_4(\Phi_\nu + \Phi^*_\nu - \Phi^T_\nu - \Phi^+_\nu)'$$

$$= T\,^1/_4(\Phi_\nu + \Phi^*_\nu - \Phi^T_\nu - \Phi^+_\nu)T^{-1} + T\partial T^{-1}/\partial x^\nu. \tag{22}$$

The only combination of Φ_ν satisfying the listed requirements is, as follows from comparison of (19) with (22),

$$\Gamma_\nu = {}^1/_4(\Phi_\nu + \Phi^*_\nu - \Phi^T_\nu - \Phi^+_\nu). \tag{23}$$

This is the relation for finding Γ_ν. Although Γ_ν and Φ_ν possess vector properties under coordinate transformations, nevertheless under T-transformations they, as is seen from (19), (21), are transformed not as algebraic functions γ^α; in this sense Γ_ν, Φ_ν are not matrix vectors. This is analogous to the fact that the Christoffel symbols are not tensors, whereas the quantity $A_{;\nu}$, constructed with the aid of the Christoffel symbols, possesses tensor transformation laws under coordinate transformations.

Let us find the explicit form of the functional dependence $\Phi_\nu(\gamma^\alpha, \gamma; \lambda^\alpha)$ and, consequently, $\Gamma_\nu(\gamma, \gamma; \lambda)$. Introduce notation for the various contractions of the matrix A: $\gamma A\gamma = \gamma^\nu A\gamma_\nu$, $SAS = S^{\alpha\beta}AS_{\alpha\beta}$; $\gamma_5\gamma A\gamma_5\gamma = \gamma_5\gamma^\nu A\gamma_5\gamma_\nu$. It is not difficult to verify by direct calculation the equalities presented in Table 1.

Table 1

A	E	γ^α	$S^{\alpha\beta}$	$\gamma_5\gamma^\alpha$	γ_5
$\gamma A\gamma$	$4E$	$-2\gamma^\alpha$	0	$2\gamma_5\gamma^\alpha$	$-4\gamma_5$
SAS	$-12E$	0	$4S^{\alpha\beta}$	0	$-12\gamma_5$
$\gamma_5\gamma A\gamma_5\gamma$	$4E$	$2\gamma^\alpha$	0	$-2\gamma_5\gamma^\alpha$	$-4\gamma_5$
$\gamma_5 A\gamma_5$	$-E$	γ^α	$-S^{\alpha\beta}$	$\gamma_5\gamma^\alpha$	$-\gamma_5$

It is easy to verify that for an arbitrary matrix A one has

$$^1/_4 E\cdot \mathrm{Sp}\, A = {^1/_{16}}A + {^1/_{16}}\gamma A\gamma - {^1/_{32}}SAS + {^1/_{16}}\gamma_5\gamma A\gamma_5\gamma - {^1/_{16}}\gamma_5 A\gamma_5. \quad (24)$$

A matrix all of whose contractions are multiples of itself has the form either $a\cdot E$, or $A^\nu\cdot\gamma_\nu$, or $B^\nu\cdot\gamma_5\gamma_\nu$, or $C^{\mu\nu}\cdot S_{\mu\nu}$, or $n\cdot\gamma_5$. With the aid of contractions it is convenient to represent the general solution of equation (11). Let $[\gamma^\sigma,\gamma_\sigma;v]_- = C_{v\tau}\gamma^\tau + D_{v\tau\rho}S^{\tau\rho} + E_{v\tau}\gamma_5\gamma^\tau + F_v\gamma_5$. Then equation (11) is satisfied if and only if $\Phi_v = {^1/_{12}}C_{v\tau}\gamma^\tau + {^1/_8}D_{v\tau\rho}S^{\tau\rho} + {^1/_4}E_{v\tau}\gamma_5\gamma^\tau + {^1/_{16}}F_v\gamma_5$.

Using Table 1 and (24), Φ_v can be represented in the form

$$\Phi_v = {^1/_{16}}[\gamma^\sigma,\gamma_\sigma;v]_- + {^{19}/_{384}}\gamma[\gamma^\sigma,\sigma;v]_-\gamma - {^1/_{96}}S[\gamma^\sigma,\gamma_\sigma;v]_-S + {^1/_{128}}\gamma_5\gamma[\gamma^\sigma,\gamma_\sigma;v]_-\gamma_5\gamma. \quad (25)$$

This expression, together with (23), gives us an explicit functional dependence of the vector Γ_v on γ^α and $\gamma^\alpha{}_{;v}$.

5. It is of interest to compare the results obtained here with analogous results within the framework of the covariant moving spin-tensor frame scheme ($^{2-7}$). The algebraic relations between spin tensors are S-invariant; contraction is permitted only over spinor indices of the same dottedness. The algebraic relations in matrix space are T-invariant, and contractions are carried out over matrix indices.

In the apparatus of spin tensors, either as an initial postulate (by analogy with $\nabla_v\gamma^{\alpha\beta} = 0$), or as a consequence, one always has $\nabla_v\gamma^\alpha = 0$. In the scheme proposed by us, generally speaking, $\nabla_v\gamma^\alpha \neq 0$.

The requirement $\nabla_v\gamma^\alpha = 0$ for all γ^α is equivalent to the condition of S-invariance of the derivative. From the conditions $\nabla\gamma = 0$ and $\nabla'\gamma' = 0$ ($\gamma' = S\gamma S^{-1}$) there follows S-invariance, and conversely, under the assumption of S-invariance, arguments analogous to those set out in Section 4 prove that always $\nabla\gamma = 0$. In our scheme the covariant derivative is only T-invariant.

Preservation of algebraic relations, including transposition and conjugation, under parallel transport of spin tensors can be achieved only by increasing the number of types of spinor indices (to 8 in the general case). In this case the affine connection is characterized by 4 spin tensors Γ_v with different spinor indices. In matrix space the operation of parallel transport does not go beyond the class of fields existing in matrix space; the covariant derivative is written with the aid of a single vector Γ_v for all matrix tensors.

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Note: Figure translations are in progress. See original paper for figures.