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Abstract

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ON ESTIMATING THE BEST APPROXIMATION OF AN INTEGRABLE FUNCTION ON THE REAL AXIS BY ENTIRE FUNCTIONS OF FINITE DEGREE

Let $W_\sigma^{(p)} (p \geqslant 1)$ be the class of entire functions $g_\sigma(z)$ of finite degree $\leqslant \sigma$, for which $g_\sigma(x) \in L_p(-\infty, \infty)$, $p \geqslant 1$. Denote by $A_\sigma(f)_p$ the best approximation of a function $f(x) \in L_p(-\infty, \infty)$ in the metric of the space $L_p(-\infty, \infty)$ by entire functions from the class $W_\sigma^{(p)}$, i.e.

$$A_\sigma(f)_p = \inf_{g_\sigma \in W_\sigma^{(p)}} \left(\int_{-\infty}^{\infty} |f(x) - g_\sigma(x)|^p \, dx \right)^{1/p},$$

and introduce for consideration the moduli of continuity

$$\omega_1(f; \delta)_p = \sup_{|h| \leqslant \delta} \left(\int_{-\infty}^{\infty} |f(x+h) - f(x)|^p \, dx \right)^{1/p},$$

$$\omega_2(f; \delta)_p = \sup_{|h| \leqslant \delta} \left(\int_{-\infty}^{\infty} |f(x+h) - 2f(x) + f(x-h)|^p \, dx \right)^{1/p}.$$

It is known that for a function $f(x) \in L_p(-\infty, \infty)$ there exist constants K_1 and K_2 satisfying the inequalities:

$$A_\sigma(f)_p \leqslant K_i \omega_i(f; \delta)_p \qquad (i = 1, 2),$$

which are analogues of the classical Jackson theorem for periodic functions (see ([1]), p. 274).

The present article is devoted* to finding the possible least values K_i^0 of the constants K_i $(i = 1, 2)$, mainly for $p = 2$. First of all, two auxiliary lemmas are proved.

Lemma 1. Let $f(x) \in L_2(-\infty, \infty)$, and let $\varphi(x)$ be its Fourier transform in the sense of $L_2(-\infty, \infty)$, i.e.

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} \varphi(t) \frac{e^{ixt} - 1}{it} \, dt, \tag{1}$$

where $\varphi(x) \in L_2(-\infty, \infty)$. Then the function

$$g_\sigma^0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \varphi(t) e^{ixt} \, dt \tag{2}$$

is an entire function from the class $W_\sigma^{(2)}$, least deviating from $f(x)$ in the metric $L_2(-\infty, \infty)$. Moreover,

$$A_\sigma(f_2) = \|f(x) - g_\sigma^0(x)\|_2 = \left(\int_{|t|>\sigma} |\varphi(t)|^2 \, dt \right)^{1/2}. \tag{3}$$

Indeed, applying the known Plancherel theorem to the function

$$f(x) - g_\sigma(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} [\varphi(x) - \psi(x)] \frac{e^{ixt} - 1}{it} \, dt,$$

* Works ([2–5]) are devoted to similar questions in the periodic case.

where $\psi(t) = 0$ for $|t| > \sigma$, we find

$$\inf_{g_\sigma \in W_\sigma^{(2)}} \|f(x) - g_\sigma(x)\|_2^2 = \inf_{\psi \in L_2(-\sigma, \sigma)} \int_{-\sigma}^{\sigma} |\varphi(t) - \psi(t)|^2 \, dt + \int_{|t|>\sigma} |\varphi(t)|^2 \, dt.$$

This implies equality (3). Applying Lemma 1 to the function

$$f^{(r)}(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{e^{ixt} - 1}{it} (it)^r \varphi(t) \, dt \in L_2(-\infty, \infty),$$

we find

$$A_\sigma(f^{(r)})_2 = \left\{ \int_{|t|>\sigma} |(it)^r \varphi(t)|^2 \, dt \right\}^{1/2} > \sigma^r \left\{ \int_{|t|>\sigma} |\varphi(t)|^2 \, dt \right\}^{1/2} = \sigma^r A_\sigma(f)_2.$$

Lemma 2. If $\varphi(t)$ is any function from $L_2(-\infty, \infty)$, then

$$\inf_{|y|\le\pi/\sigma}\int_{|t|>\sigma}|\varphi(t)|^2\cos yt\,dt<0. \tag{4}$$

Proof. Extending evenly to $(-\infty,0)$ the function

$$\varphi_\sigma(y)=-\sin\sigma y\quad\text{for } 0\le y\le\pi/\sigma;\qquad \varphi_\sigma(y)=0\quad\text{for } y>\pi/\sigma,$$

we represent it in the form of a Fourier integral

$$\varphi_\sigma(y)=\sqrt{\frac{2}{\pi}}\int_0^\infty\widetilde{\varphi}_\sigma(t)\cos yt\,dt.$$

From this we find

$$\widetilde{\varphi}(y)=\sqrt{\frac{2}{\pi}}\int_0^\infty\varphi_\sigma(y)\cos yt\,dt=\frac{1}{\sqrt{2\pi}}\frac{4\sigma}{t^2-\sigma^2}\cos^2\frac{\pi t}{2\sigma}.$$

Let us note that the function

$$F_\sigma(y)=\sqrt{\pi/2}\,|\varphi(t)|^2,\quad |t|>\sigma;\qquad F_\sigma(y)=0,\quad |t|\le\sigma,$$

is the cosine transform of the function

$$F_\sigma(y)=\int_\sigma^{+\infty}|\varphi(t)|^2\cos yt\,dt,\qquad 0\le y\le\pi/\sigma.$$

It is not difficult to show that

$$\int_0^\infty F_\sigma(y)\varphi_\sigma(y)\,dy=-\int_0^\infty\widetilde{F}_\sigma(y)\widetilde{\varphi}_\sigma(y)\,dy=2\sigma\int_\sigma^\infty|\varphi(t)|^2\frac{\cos^2\pi t/2\sigma}{t^2-\sigma^2}\,dt\ge 0. \tag{5}$$

On the other hand, if $F_\sigma(y)\ge 0$ everywhere on $[0,\pi/\sigma]$, then

$$\int_0^\infty F_\sigma(y)\varphi_\sigma(y)\,dy=-\int_0^{\pi/\sigma}F_\sigma(y)\sin\sigma y\,dy<0. \tag{6}$$

Inequality (6) contradicts (5), and, consequently, $F_\sigma(y)$ assumes both positive and negative values on $[0,\pi/\sigma]$, i.e. (4) is true. Owing to these lemmas, the following is proved.

Theorem 1. If the function $f(x) \in L_2(-\infty, \infty)$, then for the best approximation $A_\sigma(f)_2$ the inequalities

$$A_\sigma(f)_2 < \frac{1}{\sqrt{2}}\,\omega_1(f;\pi/\sigma)_2, \tag{7}$$

$$A_\sigma(f)_2 < \frac{1}{2}\,\omega_2(f;\pi/\sigma)_2 \tag{8}$$

hold.

Proof. From equality (1) we find

$$\omega_1\left(f;\frac{\pi}{\sigma}\right)_2 = \sup_{|y|\le\pi/\sigma}\|f(x+y)-f(x)\|_2 = \sqrt{2}\left\{\int_{|t|>\sigma}|\varphi(t)|^2\,dt - \inf_{0\le y\le\pi/\sigma}F_\sigma(y)\right\}.$$

Hence, by Lemmas 1 and 2, (7) follows. Moreover, for $|t| \le \pi/\sigma$,

$$\sup_{|t|\le\pi/\sigma}\|f(x+t)-2f(x)+f(x-t)\|_2^2 \ge 4\sup_{|t|\le\pi/\sigma}\int_{|u|>\sigma}|\varphi(u)|^2(1-\cos ut)^2\,du \ge$$

$$\ge 4\int_{|u|>\sigma}|\varphi(u)|^2\,du - 8\inf_{0\le y\le\pi/\sigma}F_\sigma(y).$$

Hence inequality (8) follows.

Corollary. If $f(x)$ has a derivative $f^{(r)}(x) \in L_2(-\infty, \infty)$, then the estimates

$$A_\sigma(f)_2 \le \frac{1}{\sqrt{2}\,\sigma^r}\,\omega_1(f^{(r)};\pi/\sigma)_2, \qquad A_\sigma(f)_2 \le \frac{1}{2\sigma^r}\,\omega_2(f^{(r)};\pi/\sigma)_2$$

are valid.

Along the way it is established that, for the best approximation of an entire function

$$f_\lambda(x) = \frac{1}{\sqrt{2\pi}}\int_{-\lambda}^{\lambda}\varphi(t)e^{ixt}\,dt$$

of degree λ, where $\varphi(t) \in L_2(-\lambda, \lambda)$, by entire functions from the class $W_\sigma^{(2)}$, we have

$$A_\sigma(f_\lambda)_2 = \left(\int_{\sigma<|t|\le\lambda} |\varphi(t)|^2\,dt\right)^{1/2}.$$

As an example, the function $f(x) = e^{-|x|} \in L_2(-\infty,\infty)$ with Fourier transform

$$\varphi(t) = \sqrt{\frac{2}{\pi}}\,\frac{1}{1+t^2}$$

is considered. According to Lemma 1, we have

$$A_\sigma(e^{-|x|})_2 = \left[1 - \frac{2}{\pi}\left(\operatorname{arctg}\sigma - \frac{\sigma}{1+\sigma^2}\right)\right]^{1/2} < \sqrt{\frac{2}{\sigma}}.$$

Now let $f(x)$ be an arbitrary function from the class $L_p(-\infty,\infty)$ $(1 < p \le 2)$. Then

$$F(x,a) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} f(t)e^{ixt}\,dt$$

as $a \to \infty$ converges in the mean with exponent q, where $q = p/(p-1)$. The mean limit $F(x)$, called the Fourier transform of the function $f(x)$, satisfies the inequality ([6])

$$\left(\int_{-\infty}^{\infty} |F(x)|^q\,dx\right)^{1/q} \le B_{p,q}\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p},$$

where

$$B_{p,q} = (2\pi/q)^{1/2q}(p/2\pi)^{1/2p}.$$

Moreover, for almost all x we have

$$F(x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty} f(x)\frac{e^{ixt}-1}{it}\,dt, \qquad f(x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty} F(t)\frac{e^{-itx}-1}{-it}\,dt.$$

Thanks to these assertions, the following is proved.

Theorem 2. If $f(x) \in L_p(-\infty,\infty)$ $(1 < p \le 2)$ and $f(x)$ is its Fourier transform in the sense of $L_p(-\infty,\infty)$, then the estimates

$$A_\sigma(f)_p \geq B_{p,q}^{-1}\left(\int_{|x|>\sigma} |F(x)|^q\,dx\right)^{1/q} \qquad (1/p+1/q=1),$$

$$\sup_{|y|\leq\pi/\sigma}\left(\int_{-\infty}^{\infty} |F(x)|^q \left|\sin\frac{yx}{2}\right|^q\,dx\right)^{1/q} \leq \frac{1}{2}B_{p,q}\omega_1\left(f;\frac{\pi}{\sigma}\right)_p.$$

The arguments used in proving the assertions given above can also be applied in the multidimensional case. For example, consider a function admitting the representation

$$f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixt+iy\tau}\varphi(t,\tau)\,dt\,d\tau, \tag{9}$$

and entire functions $g_{\sigma_1,\sigma_2}(x,y)\in W_{\sigma_1,\sigma_2}^{(p)}$, admitting the spectral representation

$$g_{\sigma_1,\sigma_2}(x,y)=\frac{1}{2\pi}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\psi(t,\tau)e^{ixt+iy\tau}\,dt\,d\tau,$$

where $\psi(t,\tau)\in L_2\binom{-\sigma_1,\sigma_1}{-\sigma_2,\sigma_2}$ and $\psi(t,\tau)=0$ for $(x,y)\notin\begin{bmatrix}-\sigma_1,\sigma_1\\-\sigma_2,\sigma_2\end{bmatrix}$.

Theorem 3. If the function $f(x,y)\in L_2\binom{-\infty,\infty}{-\infty,\infty}$ is defined by equality (9), then the entire function from the class $W_{\sigma_1,\sigma_2}^{(2)}$ least deviating from it in the sense of the metric $L_2\binom{-\infty,\infty}{-\infty,\infty}$ is

$$g_{\sigma_1,\sigma_2}^0(x,y)=\frac{1}{2\pi}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2} e^{ixt+iy\tau}\varphi(t,\tau)\,dt\,d\tau, \tag{10}$$

where $\varphi(x,y)$ is the Fourier transform of the function $f(x,y)$ in the sense of (9); moreover, the value of the best approximation $A_{\sigma_1,\sigma_2}(f)_2$ of the function $f(x,y)$ by entire functions from the class $W_{\sigma_1,\sigma_2}^{(2)}$ is computed from one of the equalities

$$A_{\sigma_1,\sigma_2}(f)_2=\left\{\int_{-\infty}^{\infty}\int_{|\tau|\geq\sigma_2}|\varphi(t,\tau)|^2\,dt\,d\tau+\int_{|t|>\sigma_1}\int_{|\tau|\leq\sigma_2}|\varphi(t,\tau)|^2\,dt\,d\tau\right\}^{1/2}, \tag{11}$$

or

$$A_{\sigma_1,\sigma_2}(f)_2 = \left\{ \int_{|t|>\sigma_1} \int_{-\infty}^{\infty} |\varphi(t,\tau)|^2 \, dt \, d\tau + \int_{|t|\le\sigma_1} \int_{|\tau|>\sigma_2} |\varphi(t,\tau)|^2 \, dt \, d\tau \right\}^{1/2}. \tag{12}$$

Introduce the notation:

$$A_{\sigma_1,\infty}(f)_p = \inf_{g_{\sigma_1,\infty} \in W^{(p)}_{\sigma_1,\infty}} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x,y) - g_{\sigma_1,\infty}(x,y)|^p \, dx \, dy \right\}^{1/p} \quad (p \ge 1),$$

where

$$g_{\sigma_1,\infty}(x,y) = \int_{-\sigma_1}^{\sigma_1} \int_{-\infty}^{\infty} e^{ixt+iy\tau} \varphi(t,\tau) \, dt \, d\tau$$

is an entire function in x from the class $W^{(p)}_{\sigma_1}$. The quantity $A_{\infty,\sigma_2}(f)_p$ is defined analogously, and it is proved that

$$A_{\sigma_1,\sigma_2}(f)_2 \le \left[A^2_{\sigma_1,\infty}(f)_2 + A^2_{\infty,\sigma_2}(f)_2 \right]^{1/2}.$$

Owing to this, from the inequalities

$$A^2_{\sigma_1,\infty}(f)_2 < \frac{1}{2} \omega_1^2 \left(f; \frac{\pi}{\sigma_1}; 0 \right)_2; \qquad A^2_{\infty,\sigma_2}(f)_2 < \frac{1}{2} \omega_1^2 \left(f; 0; \frac{\pi}{\sigma_2} \right)_2$$

we obtain

$$A_{\sigma_1,\sigma_2}(f)_2 < \frac{1}{\sqrt{2}} \left\{ \omega_1^2 \left(f; \frac{\pi}{\sigma_1}; 0 \right)_2 + \omega_1^2 \left(f; 0; \frac{\pi}{\sigma_2} \right)_2 \right\}^{1/2}.$$

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Note: Figure translations are in progress. See original paper for figures.