ON THE ROLE OF THE EXTREME TERMS OF A VARIATION SERIES IN THE FORMATION OF A LARGE DEVIATION OF THE SUM OF INDEPENDENT RANDOM VARIABLES

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Abstract

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MATHEMATICS

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ON THE ROLE OF THE EXTREME TERMS OF A VARIATION SERIES IN THE FORMATION OF A LARGE DEVIATION OF THE SUM OF INDEPENDENT RANDOM VARIABLES

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Let $\xi_j,\ j = 1, 2, \dots$, be independent identically distributed random variables, $M\xi_j = 0,\ D\xi_j = \sigma^2 < \infty$. Let, further,

$$\underline{\xi} = \xi_1^* \leqslant \xi_2^* \leqslant \dots \leqslant \xi_n^* = \bar{\xi} \tag{1}$$

be the variation series constructed from the realization $\xi_1, \xi_2, \dots, \xi_n$. We shall be interested in the limiting distribution law of the extreme terms of the series (1) under conditions imposed on the sum $\zeta_n = \xi_1 + \dots + \xi_n$.

At first we shall assume that the random variables ξ_j have an absolutely continuous distribution with bounded density $p(x)$, and that, as $x \to 0$,

$$p(x) \sim e^{-x^\alpha}, \qquad \alpha > 0. \tag{2}$$

Put, as usual,

$$\Phi(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w} e^{-u^2/2}\, du.$$

Let, further, the sequences A_n and B_n satisfy the equalities*

$$2n \exp\{-A_n^2/2\} = A_n\sqrt{2\pi}, \qquad B_n = A_n^{-1}, \tag{3}$$

and let the sequences a_n and b_n be such that

$$n \exp\{-a_n^\alpha\} = \alpha a_n^{\alpha-1}, \qquad b_n = \alpha a_n^{1-\alpha}. \tag{4}$$

Theorem 1. *Let in relation (2)* $\alpha > 1$. *Then*

1°. If $0 \leqslant x \leqslant n(\ln n)^{-\gamma}$, $\gamma > 1/\alpha$, *then as* $n \to \infty$

$$\mathbf{P}\{\bar{\xi} < a_n + b_{nz} \mid \zeta_n = x\} = \mathbf{P}\{\bar{\xi} < a_n + b_{nz}\} + o(1) = \exp\{-e^{-z}\} + o(1)$$

uniformly in $z,\ z \geqslant z_0 > -\infty$.

2°. If, however, $x \geqslant n(\ln n)^{\gamma_1}$, $\gamma_1 > \max(12/\alpha,\ 1/(\alpha-1))$, *then as* $n \to \infty$

$$\mathbf{P}\left\{\max_{1\leqslant j\leqslant n}\left|\xi_j - \frac{x}{n}\right|\sqrt{\frac{x^{\alpha-2}}{\alpha(\alpha-1)}} < A_n + B_{nz} \mid \zeta_n = x\right\} =$$

$$= \exp\{-e^{-z}\} + o(1)$$

uniformly in $z,\ z \geqslant z_0 > -\infty$.

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* It is easy to see that, uniformly in $z,\ -\infty < z < \infty$,

$$\mathbf{P}\left\{\max_{1\leqslant j\leqslant n}|\eta_j| < A_n + B_{nz}\right\} = \exp\{-e^{-z}\} + o(1),$$

where the η_j are independent and normally distributed with parameters $(0,1)$.

Here the sequences A_n, B_n, a_n, and b_n are defined by equations (3) and (4).

Theorem 2. Suppose that in relation (2) $0 < \alpha < 1$. Suppose further that $M|\xi_1|^k < \infty$, $k = 4 + \left[\dfrac{2\alpha-1}{1-\alpha}\right]$ (in particular, for $0 < \alpha < 1/2$ we have $k = 3$). Then:

1°. If

$$0 \le x \le (c_\alpha - \delta)\sigma^{2/(2-\alpha)}n^{1/(2-\alpha)}, \qquad c_\alpha = (2-\alpha)(2-2\alpha)^{(\alpha-1)/(2-\alpha)},$$

where $\delta > 0$ is an arbitrarily small positive number, then, as $n \to \infty$,

$$\mathbf{P}\{\bar{\xi} < a_n + b_{nz} \mid \zeta_n = x\} = \mathbf{P}\{\bar{\xi} < a_n + b_{nz}\} + o(1) = \exp\{-e^{-z}\} + o(1)$$

uniformly in $z,\ z \ge z_0 > -\infty$, and $w,\ -\infty < w < \infty$.

2°. If, however,

$$(c_\alpha - \delta)\sigma^{2/(2-\alpha)}n^{1/(2-\alpha)} < x = o\big(n^{1/(2-2\alpha)}\big),$$

then, as $n \to \infty$,

$$\mathbf{P}\{\xi^*_{n-1} < a_n + b_{nz};\ \bar{\xi} - (1-\beta)x < w\sigma\sqrt{n} \mid \zeta_n = x\} =$$

$$= \exp\{-e^{-z}\}\Phi\left(\frac{w}{\sigma_1}\right) + o(1)$$

uniformly in $z,\ z \geq z_0 > -\infty$, and $w,\ -\infty < w < \infty$.

Here the quantity β is the smaller positive root of the equation

$$\frac{n\sigma^2}{x^{2-\alpha}} = \frac{\beta(1-\beta)^{1-\alpha}}{\alpha},$$

if $0 < \alpha < 1/2$, and of the equation

$$\frac{n\sigma^2}{x^{2-\alpha}} = \frac{\beta(1-\beta)^{1-\alpha}}{\alpha}\left[1 - \sigma^2\sum_{j=0}^{k-4}(j+3)\lambda_j\left(\frac{\beta x}{n}\right)^{j+1}\right],$$

if $1/2 \leq \alpha < 1$.

Here the sequences a_n and b_n are defined by equalities (4), while $\lambda_j,\ j = 0, \ldots, k-4$, are the first coefficients of the Cramér series (see ([1]), equality (19)), and

$$\sigma_1 = \left(1 - \alpha(1-\alpha)n\sigma^2/(1-\beta)^{1+\varepsilon}x^{1+\varepsilon}\right)^{-1/2}.$$

Theorem 3. Suppose that in relation (2) $0 < \alpha < 1$. Then:

1°. If $n^{1/(2-2\alpha)} \ll x$, then, as $n \to \infty$,

$$\mathbf{P}\{\xi^*_{n-1} < a_n + b_{nz};\ \bar{\xi} - x + n\alpha\sigma^2x^{\alpha-1} < w\sigma\sqrt{n} \mid \zeta_n = x\} =$$

$$= \exp\{-e^{-z}\}\Phi(w) + o(1)$$

uniformly in $z,\ z \geq z_0 > -\infty$, and $w,\ -\infty < w < \infty$.

2°. If, however, $n^{1/(2-2\alpha)} = o(x)$, then, as $n \to \infty$,

$$\mathbf{P}\{\xi^*_{n-1} < a_n + b_{nz};\ \bar{\xi} < x + w\sigma\sqrt{n} \mid \zeta_n = x\} =$$

$$= \exp\{-e^{-z}\}\Phi(w) + o(1)$$

uniformly in $z,\ z \geq z_0 > -\infty$, and $w,\ -\infty < w < \infty$.

The theorems stated above refine the considerations on the nature of large deviations contained in the papers ([2,3]). We now briefly describe the case where the random variables are integer-valued. Suppose that instead of representation (2) the relation

$$\mathbf{P}\{\xi_1 = k\} \sim e^{-k^\alpha}, \qquad \alpha > 0. \tag{5}$$

holds.

Under condition (5), the assertions of Theorems 2 and 3 remain valid. If in relation (5) $1 < \alpha < 2$, then assertion 2° of Theorem 1 is also valid. Under the conditions of 1° of Theorem 1, there is not even an unconditional limiting distribution for $\bar{\xi}$. Concerning the case $\alpha \geq 2$, see paper ([4]).

If one does not require the existence of a density of the distribution of the random variables ξ_j and does not assume that they are integer-valued, then one has to use

to trace the limiting behavior of our order statistics under the condition that $\zeta_n > x$. For example, the following is valid.

Theorem 4. *Suppose that instead of representation (2) the relation*

$$1 - F(x) \sim x^{-\gamma}, \qquad \gamma > 2. \tag{6}$$

holds. Then, as $n \to \infty$, $x/\ln x \geqslant \sqrt{n}$

$$\mathbf{P}\{\xi^*_{n-1} < zn^{1/\gamma};\ \bar{\xi} > x + wx \mid \zeta_n > x\} = \exp\{-z^{-\gamma}\}(1+w)^{-\gamma} + o(1)$$

uniformly in z, $z \geqslant z_0 > -\infty$, *and* w, $w \geqslant 0$.

Conditions (2), (5), and (6) are one-sided in character; therefore the theorems given above (except for item 2° of Theorem 1) describe the limiting law of distribution only for the extreme right-hand terms of series (1). If the behavior of the probability $\mathbf{P}\{\xi_1 < x\}$ as $x \to -\infty$ is specified, then one can obtain an explicit expression for the limiting distribution of the statistics ξ, $\bar{\xi}$, and ξ^*_{n-1}. For example, instead of item 2° of Theorem 3 the following may be formulated.

Theorem 5. *Suppose that in representation (2)* $0 < \alpha < 1$, *and as* $x \to \infty$

$$\mathbf{P}\{\xi_1 < -x\} \sim x^{-\gamma}, \quad \gamma > 4 + \left[\frac{2\alpha - 1}{1 - \alpha}\right].$$

Then, if $n^{1/(2-2\alpha)} = o(x)$,

$$\mathbf{P}\{-yn^{1/\gamma} < \xi; \xi^*_{n-1} < a_n + b_{nz}; \bar{\xi} < x + w\sigma\sqrt{n} \mid \zeta_n = x\} = \exp\{-y^{-\gamma} - e^{-z}\}\Phi(w) + o(1)$$

uniformly in y, z, *and* w, $y \geqslant 0$, $z \geqslant z_0 > -\infty$, $-\infty < w < \infty$.

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Note: Figure translations are in progress. See original paper for figures.

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