LIMITING DISTRIBUTIONS OF FUNCTIONALS OF A SEMI-MARKOV PROCESS DEFINED ON A FIXED SET OF STATES UP TO THE MOMENT OF FIRST EXIT

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Abstract

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LIMITING DISTRIBUTIONS OF FUNCTIONALS OF A SEMI-MARKOV PROCESS DEFINED ON A FIXED SET OF STATES UP TO THE MOMENT OF FIRST EXIT

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Consider, for each $t \in (0, \infty)$, a right-continuous semi-Markov process (S.M.P.) $\varkappa_t(s) \in \{0, 1, \dots, r\}$, which is specified, following (^{1,2}), by the matrix of transition probabilities

$$F_t(i, j, u) = P\{\varepsilon_{k+1} = j,\ \tau_t(\varepsilon_k) < u \mid \varepsilon_k = i\}, \qquad i, j = 0, \dots, r,$$

where $\varepsilon_k = \varkappa_t(\theta_t(k))$, $\tau_t(\varepsilon_k) = \theta_t(k+1) - \theta_t(k)$, and $\theta_t(k)$ is the moment of the k-th jump, i.e. $\theta_t(0) = 0$, and

$$\theta_t(k) = \min\{s :\ s > \theta_t(k-1),\ \varkappa_t(s) \neq \varepsilon_{k-1}\}, \qquad k \geq 1.$$

We shall call the set of states $\langle\omega\rangle = \langle 1, 2, \dots, r\rangle$ a group if, for any $i, j \in \langle\omega\rangle$, $p_t(i, j, \langle\omega\rangle) \to 1$ as $t \to \infty$, where $p_t(i, j, \langle\omega\rangle)$ is the probability of reaching j from i before the S.M.P. leaves $\langle\omega\rangle$.

Let, for each $t \in (0, \infty)$, $f_t^{(k)}(i, x)$, $k = 0, 1, \dots$, $i \in \langle\omega\rangle$, $x \in (0, \infty)$, be a family of random variables independent of the S.M.P. $\varkappa_t(s)$, independent in the aggregate, whose distribution does not depend on the index k. Denote

$$\nu_t(j) = \min\{n :\ \varepsilon_n \in \overline{\langle\omega\rangle} \mid \varepsilon_0 = j\}, \qquad j \in \langle\omega\rangle.$$

Introduce an additive functional of the trajectory of the S.M.P. of the form

$$\xi_t(j) = \sum_{k=0}^{\nu_t(j)-1} f_t^{(k)}(\varepsilon_k, \tau_t(\varepsilon_k)), \qquad \text{if } \varepsilon_0 = j.$$

We shall study the limiting distributions of $\xi_t(j)$ under the corresponding normalization, as $t \to \infty$. Denote by $\tau_t(l, j)$ a random variable with distribution function $p_t(l, j)^{-1} F_t(l, j, u)$, $l, j \in \langle\omega\rangle$, and let

$$\gamma_t(l, j) = f_t^{(1)}(l, \tau_t(l, j)), \qquad \psi_t(l, j, \lambda) = M \exp\{i\lambda\gamma_t(l, j)\}.$$

Put

$$g_t = \sum_{k \in \langle\omega\rangle} q_t(k, \langle\omega\rangle) \sum_{i \in \overline{\langle\omega\rangle}} p_t(k, i),$$

where $q_t(k, \langle\omega\rangle)$, $k \in \langle\omega\rangle$, is the stationary distribution for the chain with matrix $\widetilde{P}(t, \langle\omega\rangle) = \|\tilde{p}_t(k, j, \langle\omega\rangle)\|$, $k, j \in \langle\omega\rangle$, and

$$\tilde{p}_t(k, j, \langle\omega\rangle) = p_t(k, j) \left(1 - \sum_{i \in \langle\omega\rangle} p_t(k, i)\right)^{-1}.$$

Here $p_t(k, j) = F_t(k, j, \infty)$, $k, j = 0, \dots, r$, are the transition probabilities for the embedded Markov chain. Suppose that there exists $\beta = \beta(g_t) \to 0$ such that, for any $l, j \in \langle\omega\rangle$,

$$\psi_t(l, j, \beta\lambda) = 1 + a_{lj}(\lambda) g_t \big(q_t(l, \langle\omega\rangle) p_t(l, j)\big)^{-1} (1 + o(1)), \tag{1}$$

where $|a_{ij}(\lambda)| \geq 0$. Note that this condition is equivalent to the condition that the quantities $\gamma_t(l, j)$ are attracted to certain infinitely divisible laws. Denote $\varphi_t(j, \lambda) = M \exp\{i\lambda\xi_t(j)\}$.

Theorem 1. *If the set $\langle\omega\rangle$ forms a group and (1) is satisfied, then, independently of the initial state $j \in \langle\omega\rangle$, as $t \to \infty$*

$$\varphi_t(j, \beta\lambda) \to \frac{1}{1 - a(\lambda)};$$

here

$$a(\lambda) = \sum_{i, j \in \langle\omega\rangle} a_{ij}(\lambda)$$

is the logarithm of the characteristic function of a certain infinitely divisible law.

Remark 1. In the case when $a_{ij}(\lambda) = c_{ij}\lambda^\alpha$, $i, j \in \langle\omega\rangle$ (c_{ij} are certain constants), the theorem has a simple probabilistic meaning:

$$\beta\xi_t(j) \xrightarrow{\text{sl}} \xi\eta^{1/\alpha}*,$$

where ξ is a stable distribution with exponent α, $0 < \alpha \leq 2$, η is an exponential distribution, and ξ and η are independent.

Remark 2. If, for the quantities $\gamma_t(k,j)M_t^{-1}$, $k, j \in \langle\omega\rangle$, condition (1) is satisfied, where $a_{kj}(\lambda) = i\lambda c_{kj}$, and, in addition, the quantities $\gamma_t(k,j)$ have an n-th moment ($n \geq 1$), with

$$M_t^{-l}g_t^{l-1}q_t(k,\langle\omega\rangle)p_t(k,j)M\gamma_t(k,j)^l = o(1),$$

$$1 < l \leq n, \qquad k, j \in \langle\omega\rangle,$$

then under the conditions of Theorem 1

$$M_t^{-1}g_t\xi_t(j) \xrightarrow{\text{sl}} \eta$$

and all moments up to order n, inclusive, converge. Here

$$M_t = \sum_{i\in\langle\omega\rangle} q_t(i,\langle\omega\rangle)Mf_t^{(1)}(i,\tau_t(i)),$$

η is exponential and $M\eta = 1$. In particular, if we set $f_t^{(k)}(i,x) = 1$ or $f_t^{(k)}(i,x) = x$, $i \in \langle\omega\rangle$, we obtain that

$$g_t\nu_t(j) \xrightarrow{\text{sl}} \eta, \qquad m_t^{-1}g_t\Omega_t(i) \xrightarrow{\text{sl}} \eta,$$

where $\Omega_t(j)$ is the time spent by the s.m.p. in the group $\langle\omega\rangle$ before exit, under the condition that $\varepsilon_0 = j \in \langle\omega\rangle$, and

$$m_t = \sum_{i\in\langle\omega\rangle} q_t(i,\langle\omega\rangle)M\tau_t(i).$$

This agrees with the results of ([3],[4]).

Let us now assume that the quantities $\gamma_t(k,j)$ have a first moment and that there exists $\beta = \beta(g_t)$ such that, for all $k, j \in \langle\omega\rangle$,

$$\psi_t(k,j,\beta\lambda) = 1 + i\lambda m_t(k,j)\beta + a_{kj}(\lambda)g_t\frac{1+o(1)}{q_t(k,\langle\omega\rangle)p_t(k,j)}. \tag{2}$$

Theorem 2. *If the set $\langle\omega\rangle$ forms a group and (2) is satisfied, then, independently of the initial state $l \in \langle\omega\rangle$, the distribution of the random variable*

$$\alpha_t(\xi_t(l) - M_t\nu_t(l))$$

converges weakly to the distribution with characteristic function

$$\frac{1}{1 - c_1 a(\lambda) + c_2\lambda^2},$$

* In the sense of weak convergence of distribution functions.

where $\alpha_t = \min\{\sqrt{g_t}M_t^{-1}, \beta\}$; $a(\lambda)$ is defined as in Theorem 1, and $c_1 \geqslant 0$ and $c_2 \geqslant 0$ are certain constants, with, if

$$\beta^2 M_t^2 g_t^{-1} \to \mu,$$

then for $\mu = \infty$, $c_1 = 0$, and for $\mu = 0$, $c_2 = 0$.

In conclusion we indicate an effective algorithm which, in a finite number of steps, not exceeding $r - 1$, makes it possible to check whether the given set $\langle\omega\rangle$ will form a group.

Call a transition from i to j "proper" if $P_t(i,j) \nrightarrow 0$. We perform the following operation with our embedded chain: at the first step we carry out all "proper" transitions. We obtain a certain directed graph. It is obvious that connected subsets from which there are no transitions form groups. Replace each such group $\langle d\rangle$ by one state with transition probabilities

$$p_t(\langle d\rangle, j) = \left(\sum_{k\in\langle d\rangle} q_t(k, \langle d\rangle) \sum_{l\in\langle d\rangle} p_t(k,l)\right)^{-1} \sum_{i\in\langle d\rangle} q_t(i, \langle d\rangle) p_t(i,j);$$

$$p_t(j, \langle d\rangle) = \sum_{k\in\langle d\rangle} p_t(j,k); \qquad j \in \overline{\langle d\rangle}.$$

Thus, after the first step we obtain a new chain with a smaller number of states. With it we perform the same operation. Then it is clear that $\langle\omega\rangle$ forms a group if and only if all of $\langle\omega\rangle$ can be reduced to a single state.

It is of interest to note that the types of distributions found were indicated earlier in ([5],[6]) and, through the construction of $\xi_t(j)$, are naturally connected with the problem considered there.

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Note: Figure translations are in progress. See original paper for figures.