THEOREMS OF PHRAGMÉN-LINDELÖF TYPE FOR SOLUTIONS OF ELLIPTIC EQUATIONS OF HIGH ORDER

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Abstract

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THEOREMS OF PHRAGMÉN–LINDELÖF TYPE FOR SOLUTIONS OF ELLIPTIC EQUATIONS OF HIGH ORDER

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Two types of generalizations of the classical Phragmén–Lindelöf theorem for analytic functions of a complex variable to harmonic functions are possible. In the first of them, with an analytic function $f(z)$ one associates the harmonic function $\ln|f(z)|$. This yields theorems of the following type: on the boundary of a certain unbounded domain the Dirichlet data of a harmonic function are nonpositive; then either this function is nonpositive everywhere in the domain, or at infinity it grows no more slowly than at a certain rate depending on the shape of the domain (for a strip this is an exponential). In the second generalization, the function $f(z)$ is associated with the harmonic function $\operatorname{Re} f(z)$. Here theorems of the following kind arise: if on the boundary of an unbounded domain the Cauchy data of a harmonic function are bounded in modulus, then this function either is uniformly bounded in modulus everywhere in this domain, or at infinity grows no more slowly than at a certain rate depending on the domain (for a strip this is an iterated exponential).

In the present note we shall deal with a generalization of the second type. For harmonic functions of many independent variables, as well as for solutions of a number of particular types of elliptic equations, this question has been studied by M. A. Evgrafov, I. S. Arshon, M. A. Pak, G. A. Dzhafarli ([1–7]) and others.

In the note the case of a general linear elliptic equation with variable coefficients of arbitrary order without multiple characteristics is considered (more precisely, subject to Hörmander’ s condition, which will be formulated below).

Let

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

be a uniformly elliptic operator defined in a domain $\Omega \subset \mathbf{R}^n$. The coefficients (generally speaking, complex) of the derivatives of highest order are from C_1, while the remaining ones satisfy a Hölder condition and are bounded in Ω.

We shall say that the equation

$$P(x, D)u = 0 \tag{1}$$

satisfies Hörmander' s condition ([8]) if, for any real vector $N \neq 0$ and any real vector ξ not proportional to N, the equation $P_m(x, \xi + i\tau N) = 0$, where

$$P_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha,$$

has no double real roots τ.

We shall assume that equation (1) satisfies this Hörmander condition.

Theorem 1. *Let Ω be a cylinder defined by the inequalities*

$$\sum_{i=2}^{n} x_i^2 \leq h^2, \qquad x_1 \leq x_1^0,$$

and let S be its lateral surface. Let $u(x)$ be a solution in Ω of equation (1), satisfying the conditions

$$\left|\partial^k u/\partial n^k\right|\big|_S < 1, \qquad k = 0, \dots, m-1$$

($\partial/\partial n$ is differentiation along the normal to the surface S).

Then either $u(x)$ is uniformly bounded in Ω, or

$$\overline{\lim_{x_1 \to \infty}} \left[|u(x)|/\exp(\exp Cx_1)\right] > 0, \tag{2}$$

where $C > 0$ is a constant depending on the equation and on h.

Proof. For an arbitrary $a \geq x_1^0 + 1$, denote by Π_a and Π'_a the cylinders defined, respectively, by the inequalities

$$\sum_{i=2}^{n} x_i^2 \leq h^2, \quad a - 1 \leq x_1 \leq a + 2 \quad \text{and} \quad \sum_{i=2}^{n} x_i^2 \leq h^2, \quad a \leq x_1 \leq a + 1.$$

Let S_a be the lateral surface of the cylinder Π_a.

From the results of M. M. el Borai ([10]) it follows that there exist constants $\varepsilon_0 > 0$ and μ, $0 < \mu < 1$, depending on the equation and on h, such that, if ε, $0 < \varepsilon < \varepsilon_0$, is an arbitrary number and $v(x)$ is a solution of equation (1) in Π_a satisfying the conditions

$$|v| \leq 1 \text{ in } \Pi_a, \qquad \left|\partial^k v/\partial n^k\right|\big|_{S_a} < \varepsilon, \quad k = 0, \dots, m-1, \tag{3}$$

then

$$|v| < \varepsilon^{\mu} \text{ in } \Pi'_a. \tag{4}$$

Denote

$$M_a = \max_{x \in \Pi_a} |u(x)| \quad \text{and} \quad M'_a = \max_{x \in \Pi'_a} |u(x)|.$$

Put $M = \max(1/\varepsilon_0, 1)$ and suppose that $u(x)$ is unbounded in the cylinder Ω. Then one can find a number $a \geq x_1^0 + 2$ such that

$$M'_a > M, \qquad M'_a > M'_{a-1}. \tag{5}$$

Consider in the cylinder Π_a the function $v(x) = u(x)/M_a$. It is a solution of equation (1), and for it we have $|v(x)| \leq 1$ in Π_a,

$$\left|\partial^k v/\partial n^k\right|\big|_{S_a} < 1/M_a, \quad k = 0, \dots, m-1.$$

Consequently, by (3) and (4),

$$|v| < (1/M_a)^{\mu} \text{ in } \Pi'_a,$$

i.e.

$$M_a > (M'_a)^{1/(1-\mu)}. \tag{6}$$

Since $M_a = \max(M'_{a-1},\, M'_a,\, M'_{a+1})$, it follows from (5) and (6) that $M_a = M'_{a+1}$, and thus

$$M'_{a+1} > (M'_a)^{1/(1-\mu)}.$$

Now we may repeat the argument, replacing a by $a+1$, then by $a+2$, and so on. We obtain

$$M'_{a+k} > (M'_a)^{(1/(1-\mu))^k} = \exp\left[\ln M'_a \exp\left(k \ln \frac{1}{1-\mu}\right)\right],$$

whence inequality (2) follows, where as C one may take any number less than $\ln[1/(1-\mu)]$.

Theorem 2. Let Ω be the same cylinder as in the preceding theorem. Let $u(x)$ be a solution of equation (1) in Ω. Let $\alpha > 0$ be an arbitrary constant, and suppose the inequalities

$$\overline{\lim_{x_1\to\infty}}\left[\left|\frac{\partial^k u(x)}{\partial n^k}\right|\Big|_S \Big/ \exp(-\exp\alpha x_1)\right] < \infty, \qquad k = 0, \dots, m-1.$$

Then either

$$\overline{\lim_{x_1\to\infty}}\,[|u(x)|/\exp(-\exp\alpha x_1)] < \infty,$$

or

$$\overline{\lim_{x_1\to\infty}}\,[|u(x)|/\exp(C\exp\alpha x_1)] > 0,$$

where $C > 0$ is a constant depending on the equation, on α, and on h.

The proof is similar to the proof of the preceding theorem.

Theorem 3. Let Ω be a ball with its center removed,

$$\Omega = \{x \mid 0 < |x| \le R\}.$$

Let K be the cone defined by the inequalities

$$\sum_{i=2}^{n} x_i^2 < a^2 x_1^2, \qquad a \ne 0, \qquad x_1 > 0.$$

Let $u(x)$ be a solution of equation (1) in Ω, bounded in $\Omega \cap K$. Then either $u(x)$ is bounded everywhere in Ω, or

$$\overline{\lim_{|x|\to 0}}\,[|u(x)|/\exp(1/|x|^C)] > 0,$$

where $C > 0$ is a constant depending on the equation.

Proof. We exclude from Ω the narrower cone K'

$$K' = \left\{x \,\middle|\, \sum_{i=2}^{n} x_i^2 < \frac{a^2}{4}x_1^2,\ x_1 > 0\right\}$$

and denote the remaining set by Ω'. We map Ω' onto the cylinder

$$\hat{\Omega} = \left\{y \,\middle|\, \sum_{i=2}^{n} y_i^2 \le 1,\ y_1 \ge \ln\frac{1}{R}\right\}$$

in Euclidean space $(y_1, \dots, y_n)$ by means of the following transformation. Let $\hat{K}_{n-1}$ be the ball

$$\sum_{i=2}^{n} y_i^2 \le 1$$

on the hyperplane $y_1 = 0$. Let S_n be the unit n-dimensional sphere $|x| = 1$. Put $S_n' = S_n \setminus K'$. Let $f : S_n' \to \hat{K}_{n-1}$ be some fixed m-times continuously differentiable diffeomorphism such that it maps $S_n \setminus K$ into the ball concentric with $\hat{K}_{n-1}$ of half the radius.

Let $x \in \Omega'$. Put $y_1 = \ln 1/|x|$ and $(y_2, \dots, y_n) = f(x/|x|)$. This transformation carries equation (1) into the equation $\hat{P}(y, D_y)u = 0$, uniformly elliptic in the cylinder $\hat{\Omega}$ and with the same properties of the coefficients as equation (1). Its solution $\hat{u}(y) = u(x)$ will be uniformly bounded in the cylindrical layer

$$\frac{1}{4} < \sum_{i=2}^{n} y_i^2 < 1,$$

and therefore ([9]) there exists a constant $\hat{M} > 0$ such that on the lateral surface $\hat{S}$ of the cylinder

$$\hat{\Omega} = \left\{ y \middle| \sum_{i=2}^{n} y_i^2 \le \frac{1}{2},\ y_1 \ge \ln \frac{1}{R} \right\}$$

one has $|\partial^k \hat{u}/\partial n| < \hat{M}$, $k = 0, \dots, m-1$. Applying Theorem 1 to u in $\hat{\Omega}$ and then making the inverse transformation to the variables x, we obtain inequality (7).

Similarly, from Theorem 2, using the uniqueness theorem from ([11]), the following theorem is obtained:

Theorem 4. Let Ω and K have the same meaning as in the preceding theorem. Let $\alpha > 0$ be an arbitrary number. Let $u(x) \not\equiv 0$ be a solution of equation (1) in Ω, and let the equation itself be defined in $\overline{\Omega}$. Suppose

$$\overline{\lim_{\substack{x \in K \\ |x| \to 0}}} [|u(x)| / \exp(-1/|x|^\alpha)] < \infty.$$

Then

$$\overline{\lim_{\substack{x \in \Omega \\ |x| \to 0}}} [|u(x)| / \exp(C/|x|^\alpha)] > 0,$$

where $C > 0$ is a constant depending on the equation and on α.

In a similar way one proves a theorem generalizing Theorem 3:

Theorem 5. Let Ω and K be the same as above. Let $u(x)$ be a solution of equation (1) in Ω, and suppose that for some number C_1 the inequality

$$\overline{\lim_{\substack{x\in K\\|x|\to 0}}}\,[\,|u(x)|/|x|^{C_1}\,] < \infty.$$

is satisfied. Then either

$$\overline{\lim_{\substack{x\in \Omega\\|x|\to 0}}}\,[\,|u(x)|/|x|^{C_1}\,] < \infty,$$

or

$$\overline{\lim_{\substack{x\in \Omega\\|x|\to 0}}}\,[\,|u(x)|/\exp(1/|x|^{C})\,] > 0,$$

where $C > 0$ is a constant depending on the equation.

Instead of estimating u by the maximum of its modulus, one can pass to an estimate in L_2. Then it is not necessary to require Hölder continuity of the coefficients of the lower-order terms. Their measurability and boundedness are sufficient. In this form, for example, Theorem 5 may be reformulated as follows:

Theorem 6. Let Ω and K be the same as in Theorem 5. Put

$$M(r) = \left(\int_{|x|<r} u^2\,dx\right)^{1/2}; \qquad M_K(r) = \left(\int_{\substack{|x|<r\\x\in K}} u^2\,dx\right)^{1/2}.$$

Let $u(x) \in H_m^{\mathrm{loc}}$ be a solution of equation (1) with the new conditions on the lower-order coefficients in Ω. Suppose that for some number $C_1 > 0$ the inequality

$$\overline{\lim_{|x|\to 0}}\,[\,M_k(|x|)/|x|^{C_1}\,] < \infty.$$

is satisfied. Then either

$$\overline{\lim_{|x|\to 0}}\,[\,M(|x|)/|x|^{C_2}\,] < \infty,$$

or

$$\overline{\lim_{|x|\to 0}}\,[\,M(|x|)/\exp(1/|x|^{C})\,] > 0,$$

where $C > 0$ is a constant depending on the equation, and $C_2 > 0$, moreover, also on C_1.

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Note: Figure translations are in progress. See original paper for figures.