INTEGRAL REPRESENTATIONS OF SOLUTIONS

THEORY OF ELASTICITY

1970

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Abstract

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UDC 539.4

THEORY OF ELASTICITY

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INTEGRAL REPRESENTATIONS OF SOLUTIONS

IN THE THEORY OF SHALLOW SHELLS

(Presented by Academician Yu. N. Rabotnov on 15 VII 1968)

Integral representations of solutions in the theory of shallow shells, under the condition that the coefficients of these equations are analytic functions of the coordinates, are contained in ([1]). The kernels entering the integral representations are specified by equations of Volterra type in a two-dimensional domain. In the present paper a general solution of the equations of the technical theory of shallow shells is obtained; moreover, the kernels entering the integral representations are written in explicit form.

1. The system of differential equations of the technical theory of shallow shells has the form

$$\nabla^2\nabla^2 w = \frac{1}{D}\nabla_k^2 U, \qquad \nabla^2\nabla^2 U = -Eh\nabla_k^2 w,$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \nabla_k^2 = \frac{1}{R}\frac{\partial^2}{\partial x^2} + \frac{1}{R_1}\frac{\partial^2}{\partial y^2}, \qquad D = \frac{Eh^3}{12(1-\mu^2)}; \quad (1)$$

U and w are the stress function and the deflection in the shell; E, μ, and h are Young' s modulus, Poisson' s ratio, and the shell thickness; R and R_1 are the corresponding radii of curvature.

We write the system (1) in the form

$$\frac{\partial^4 F}{\partial z^2 \partial \zeta^2} - \frac{\partial^2 F}{\partial z^2} - 2\delta \frac{\partial^2 F}{\partial z \partial \zeta} - \frac{\partial^2 F}{\partial \zeta^2} = 0, \quad (2)$$

where

$$z = \frac{\beta\sqrt{i}}{a}(x+iy), \qquad \zeta = \frac{\beta\sqrt{i}}{a}(x-iy), \qquad F(z,\zeta) = U + i\frac{w}{e^*}, \qquad \delta = \frac{1+\alpha}{1-\alpha},$$

$$\beta = \frac{1}{4}\sqrt{\varepsilon(1-\alpha)}, \qquad \alpha = \frac{R}{R_1} \ll 1, \qquad \varepsilon = \frac{a^2}{Rh}\sqrt{12(1-\mu^2)},$$

$$e^* = \frac{1}{Eh^2}\sqrt{12(1-\mu^2)},$$

a is a certain characteristic linear dimension.

2. Represent the required analytic function $F(z,\zeta)$ in the form

$$F(z,\zeta) = \sum_{k=0}^{\infty}\left\{\frac{(z-z_0)^k}{k!}v_k(\zeta) + \frac{(\zeta-\zeta_0)^k}{k!}\mu_k(z)\right\}, \tag{3}$$

where $v_k(\zeta)$ and $\mu_k(z)$ are analytic functions of their arguments.

Substituting expression (3) into the differential equation (2), we obtain the relations

$$v''_{k+2}(\zeta) - v_{k+2}(\zeta) - 2\delta v'_{k+1}(\zeta) - v''_k(\zeta) = 0 \quad (k = 0, 1, ...),$$

$$\mu''_{k+2}(z) - \mu_{k+2}(z) - 2\delta\mu'_{k+1}(z) - \mu''_k(z) = 0. \tag{4}$$

Thus, the matter reduces to integrating infinite systems of ordinary differential equations of the form (4). Using the operator method, we obtain the operator equation corresponding to the first system of equations (4):

$$F_{k+2}(p,\delta) - 2g\delta F_{k+1}(p,\delta) - gpF_k(p,\delta) = 0, \qquad g = p/(p^2-1). \tag{5}$$

We solve the difference equation (5) by the usual methods ([2]). It has the form

$$F_{k+2}(p,\delta) = gpf_k(p,\delta)F_0(p) + f_{k+1}(p,\delta)F_1(p) \qquad (k = 0, 1, ...), \tag{6}$$

where

$$f_k(p,\delta) = (\lambda_1^{k+1} - \lambda_2^{k+1})/(\lambda_1 - \lambda_2), \qquad \lambda_{1,2} = g\left(\delta \pm \sqrt{\delta^2 + p^2 - 1}\right).$$

The functions $f_k(p,\delta)$ are polynomials in g and δ:

$$f_k(p,\delta) = g^k \sum_{j=0}^{[k/2]} \frac{(k-j)!(2\delta)^{k-2j}}{(k-2j)!\,j!}(p^2-1)^j. \tag{7}$$

Relation (6) expresses all $F_{k+2}(p,\delta)$ in terms of two arbitrary operator functions $F_0(p)$ and $F_1(p)$. Put

$$F_0(p) = [a_0/2 + F_0^*(p)]\, g,$$

$$F_1(p) = [a_0/2 + F_0^*(p)]\, f_1(p,\delta)g + [a_1/2 + F_1^*(p)]\, f_0(p,\delta)g/p, \tag{8}$$

where a_0 and a_1 are arbitrary constants. After substituting expressions (8) into formula (6), we obtain

$$F_k(p,\delta) = \left[\frac{a_0}{2} + F_0^*(p)\right] g f_k(p,\delta) + \left[\frac{a_1}{2} + F_1^*(p)\right] \frac{g}{p} f_{k-1}(p,\delta) \tag{9}$$

$$(k = 0, 1, ...),$$

moreover, as is clear from (6), $f_{-1}(p,\delta) \equiv 0$. Returning to the originals, we have

$$v_k(\zeta) = \frac{a_0}{2} g_k'(\zeta-\zeta_0) + \frac{a_1}{2} g_{k-1}(\zeta-\zeta_0) +$$

$$+ \int_{\zeta_0}^{\zeta} [g_k'(\zeta-\tau)v_0^*(\tau) + g_{k-1}(\zeta-\tau)v_1^*(\tau)]\, d\tau, \tag{10}$$

where $g_k'(\zeta) \doteq g f_k(p,\delta)$, $g_k(0) = 0$. An analogous formula, obviously, also holds for the function $\mu_k(z)$. The functions $g_k(\zeta)$ satisfy the system of equations

$$g_{k+2}''(\zeta) - g_{k+2}(\zeta) - 2\delta g_{k+1}'(\zeta) - g_k''(\zeta) = 0 \qquad (k = 0, 1, ...),$$

$$g_1''(\zeta) - g_1(\zeta) - 2\delta g_0'(\zeta) = 0, \qquad g_0''(\zeta) - g_0(\zeta) = 0. \tag{11}$$

We have

$$g_0(\zeta) = \operatorname{sh}\zeta, \qquad g_1(\zeta) = \delta\zeta\, \operatorname{sh}\zeta,$$

$$g_{2k+2}(\zeta) = \operatorname{sh}\zeta + \int_0^{\zeta} \operatorname{sh}(\xi-\tau)\left[2\delta g_{2k+1}'(\tau) + g_{2k}'(\tau)\right] d\tau, \tag{12}$$

$$g_{2k+3}(\zeta) = \int_0^{\zeta} \operatorname{sh}(\zeta-\tau)\left[2\delta g_{2k+2}' + g_{2k+1}'\right] d\tau.$$

It is easy to obtain an explicit expression for $g_k(\zeta)$:

$$g_{2k}(\zeta) = \sum_{s=0}^{\infty} \frac{\xi^{2s+1}}{(2s+1)!}\, a_{k,s}(\delta), \qquad g_{2k+1}(\zeta) = \sum_{s=0}^{\infty} \frac{\xi^{2s+2}}{(2s+2)!}\, b_{k,s}(\delta). \tag{13}$$

where

$$a_{k,s} = (k+s)! \sum_{j=0}^{s_k} \frac{(2\delta)^{2j}}{(k-j)!(s-j)!(2j)!},$$

$$b_{k,s} = (k+s+1)! \sum_{j=0}^{s_k} \frac{(2\delta)^{2j+1}}{(k-j)!(s-j)!(2j+1)!},$$

$$s_k = \begin{cases} s, & \text{for } s \leqslant k, \\ k, & \text{for } s > k; \end{cases} \qquad a_{k,s} = a_{s,k}, \quad b_{k,s} = b_{s,k}.$$

3. Substitute expressions of the type (10) for $v_k(\zeta)$ and $\mu_k(z)$ into formula (3). We readily obtain

$$\begin{aligned} F(z,\zeta) = \sum_{k=0}^{1} a_k G_k(z_0,\zeta_0,z,\zeta) + \sum_{k=0}^{1} \int_{\zeta_0}^{\zeta} G_k(z_0,\tau,z,\zeta) v_k^*(\tau)\, d\tau \\ + \sum_{k=0}^{1} \int_{z_0}^{z} G_k(t,\zeta_0,z,\zeta) \mu_k^*(t)\, dt, \end{aligned} \tag{14}$$

where

$$G_1(t,\tau,z,\zeta) = \sum_{k=0}^{\infty} \frac{(z-t)^{k+1}}{(k+1)!}\, g_k(\zeta-\tau), \qquad G_0(t,\tau,z,\zeta) = \frac{\partial^2 G_1}{\partial t\, \partial \tau};$$

$v_k^*(\zeta)$ and $\mu_k^*(z)$ $(k = 0, 1)$ are arbitrary analytic functions of their arguments, and $G_1(t,\tau,z,\zeta)$ is the Riemann function of equation (2).

The representations (14) give all regular solutions of the differential equation (2).

Received
15 IV 1968

References

1. I. N. Vekua, *New Methods for Solving Elliptic Equations*, Moscow–Leningrad, 1948.

2. A. O. Gelfond, *Calculus of Finite Differences*, Moscow, 1952.

Note: Figure translations are in progress. See original paper for figures.