SovietRxiv

Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

MATHEMATICS

1970

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-197001.67816

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1970, Volume 194, No. 3

UDC 517.946

MATHEMATICS

A. K. GUSHCHIN, V. P. MIKHAILOV

ON THE STABILIZATION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR A PARABOLIC EQUATION

(Presented by Academician A. N. Tikhonov on March 2, 1970)

Let $u(x,t)$ be the solution of the Cauchy problem for the parabolic equation

$$p(x)u_t(x,t) = \Delta u(x,t), \qquad x = (x_1, \dots, x_n) \in R_n, \quad t > 0, \tag{1}$$

$$u(x,0) = \varphi(x), \tag{2}$$

where $p(x) \geq a > 0$.

Suppose that the functions $p(x)$ and $\varphi(x)$ satisfy the following conditions:

$$\begin{gathered} p(x) \in B^{\alpha}_{[n/2]-1}(R_n) \quad \text{for } n \geq 4, \\ p(x) \in B^{\alpha}_{0}(R_n) \quad \text{for } n < 4, \end{gathered} \tag{3}$$

$$\varphi(x) \in B^0_0(R_n). \tag{4}$$

By the class $B^{\alpha}_k(R_n)$, where k is an integer, we mean the set of all bounded functions satisfying a Hölder condition of order $\alpha > 0$ in R_n, together with all continuous derivatives up to order k.

The purpose of this note is to prove the following theorem.

Theorem. If the functions $p(x)$ and $\varphi(x)$ satisfy conditions (3), (4), and

$$\frac{n}{\omega_n R^n} \int_{|x-y| \leq R} |p(y) - b| \, dy \to 0 \quad \text{as } R \to \infty \tag{5}$$

SovietRxiv

uniformly with respect to $x \in R_n$, for some constant b

$$\left(\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}\right),$$

then a necessary and sufficient condition for the existence of the limit

$$\lim_{t\to\infty} u(x,t) = A \tag{6}$$

at some $x \in R_n$ is the existence of the limit

$$\frac{n}{\omega_n R^n}\int_{|x-y|\le R} \varphi(y)\,dy \to A \quad \text{as } R\to\infty. \tag{7}$$

In order that condition (6) hold uniformly on any compact set $D \subset R_n$ (or uniformly in R_n), it is necessary and sufficient that condition (7) hold uniformly on any compact set $D \subset R_n$ (respectively, uniformly in R_n).

We note that from the results of the note ([1]) it follows that in the case under consideration ($\varphi(x) \in B_0^0(R_n)$), A is necessarily a constant.

It is easy to see that condition (5) is satisfied, in particular, if

$$\lim_{|y|\to\infty} (p(y) - b) = 0$$

or if

$$p(y) - b \in L_r(R_n) \tag{5'}$$

for some $r \ge 1$.

Without loss of generality, one may assume that in (5) $b = 1$. Denote by $v(x,t)$ the solution of the heat equation with initial function $p(x)\varphi(x)$,

$$v(x,t) = \frac{1}{(2\sqrt{\pi t})^n}\int_{R_n} \exp\left(-\frac{|x-y|^2}{4t}\right) p(y)\varphi(y)\,dy.$$

Lemma 1. Let $p(x) \in B_0^0(R_n)$, and let conditions (4) and (5) be satisfied. Then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (u(x,t) - v(x,t))\,dt = 0 \tag{8}$$

uniformly with respect to $x \in R_n$.

SovietRxiv

Let $\tilde{u}(x,\lambda)$ and $\tilde{v}(x,\lambda)$ be the Laplace transforms of the functions $u(x,t)$ and $v(x,t)$. Since $u(x,t)$ and $v(x,t)$ are bounded for $t \geq 0$, $x \in R_n$, it follows that $\tilde{u}(x,\lambda)$ and $\tilde{v}(x,\lambda)$ are analytic in λ and bounded in x for $\operatorname{Re}\lambda > 0$. The function $\tilde{u}(x,\lambda)$ satisfies the differential equation

$$-\Delta\tilde{u}(x,\lambda) + \lambda\tilde{u}(x,\lambda) = p(x)\varphi(x) - \lambda q(x)\tilde{u}(x,\lambda), \tag{9}$$

where $p(x) = 1 + q(x)$. Equation (9) in the class $B_0^0(R_n)$ is equivalent to the integral equation

$$\tilde{u}(x,\lambda) = -\mathcal{L}(\lambda)\tilde{u}(x,\lambda) + \tilde{v}(x,\lambda), \tag{10}$$

where the operator $\mathcal{L}(\lambda)$ is defined as follows:

$$\mathcal{L}(\lambda)f(x,\lambda) = \frac{\lambda^{(n+2)/4}}{(2\pi)^{n/2}} \int_{R_n} \frac{K_{n/2-1}(\sqrt{\lambda}\,|x-y|)}{|x-y|^{n/2-1}} q(y)f(y,\lambda)\,dy,$$

$$\tilde{v}(x,\lambda) = \frac{\lambda^{(n-2)/4}}{(2\pi)^{n/2}} \int_{R_n} \frac{K_{n/2-1}(\sqrt{\lambda}\,|x-y|)}{|x-y|^{n/2-1}} p(y)\varphi(y)\,dy.$$

The function $K_\nu(z)$ is the Macdonald cylindrical function.

It can be proved that the operator $\mathcal{L}(\lambda)$ is a bounded operator from $B_0^0(R_n)$ into $B_0^0(R_n)$ ($B_0^0(R)$ is a Banach space with norm

$$\|f\|_{B_0^0(R_n)} = \sup_{x\in R_n} |f(x)|$$

and

$$\|\mathcal{L}(\lambda)\|_{B_0^0(R_n)} = o(1) \quad \text{as } \lambda \to 0, \tag{11}$$

when $|\arg\lambda| \leq \pi - \sigma$ for any $\sigma > 0$. From the boundedness of the functions $p(x)$ and $\varphi(x)$ it follows that

$$\|\tilde{v}(x,\lambda)\|_{B_0^0(R_n)} \leq \frac{C}{|\lambda|} \quad \text{for } |\arg\lambda| \leq \pi - \sigma. \tag{12}$$

Consequently, from (10) and (12) we have

$$\|\tilde{u}(x,\lambda)\|_{B_0^0(R_n)} \leq \|\mathcal{L}(\lambda)\|\,\|\tilde{u}(x,\lambda)\|_{B_0^0(R_n)} + \frac{C}{|\lambda|},$$

SovietRxiv

or, by virtue of (11),

$$\|\tilde{u}(x,\lambda)\|_{B_0^0(R_n)} \le \frac{C_1}{|\lambda|} \quad \text{for } |\arg\lambda| \le \pi - \sigma,\ |\lambda| < \varepsilon,$$

where $\varepsilon > 0$ is a sufficiently small number. From (10) and (11) it then follows immediately that

$$\|\tilde{u}(x,\lambda) - \tilde{v}(x,\lambda)\|_{B_0^0(R_n)} = o(1/\lambda)$$

for $|\arg\lambda| \le \pi - \sigma,\ |\lambda| < \varepsilon$.

Relation (8) then follows from the Tauberian theorem of Wiener ([2]).

Lemma 2. *If the functions $p(x)$ and $\varphi(x)$ satisfy conditions (3), (4), (5), then*

$$\|\partial u(x,t)/\partial t\|_{B_0^0(R_n)} \le C/t \tag{13}$$

for $t \ge t_0 > 0$.

The function $w(x,t) = \partial u(x,t)/\partial t$ is, for $t > \delta$ $(0 < \delta < t_0)$, a solution of equation (1) with the initial condition

$$w(x,t)\big|_{t=\delta} = u_t(x,\delta) \equiv \psi(x). \tag{14}$$

From condition (3) it follows that

$$\psi(x) \in B^0_{[n/2]}(R_n) \quad \text{for } n \ge 4; \qquad \psi(x) \in B^0_2(R_n) \quad \text{for } n < 4. \tag{15}$$

The Laplace transform $\widetilde{w}(x,\lambda)$ of the function $w(x,t)$ satisfies equations (9) and (10), in which the function $\varphi(x)$ is replaced by the function $\psi(x)$.

Simultaneously with problem (1), (14), consider the following Cauchy problem for the hyperbolic equation:

$$p(x)z_{tt}(x,t) - \Delta z(x,t) = 0, \qquad t > \delta,$$

$$z(x,\delta) = 0, \qquad z_t(x,\delta) = \psi(x).$$

From relations (15) and the Sobolev embedding theorems ([3]) it follows that

$$|z(x,t)| \le Ct^m$$

SovietRxiv

for some $m > 0$, with a constant C independent of x, t. Therefore the Laplace transform $\tilde{z}(x, \lambda)$ of the function $z(x, t)$, satisfying the equation

$$-\Delta \tilde{z}(x, \lambda) + \lambda^2 \tilde{z}(x, \lambda) = p(x)\psi(x) - \lambda^2 q(x)\tilde{z}(x, \lambda), \tag{16}$$

is an analytic function of λ and bounded in x for $\operatorname{Re}\lambda > 0$, and $|\tilde{z}(x, \lambda)| \to 0$ as $|\lambda| \to \infty$ uniformly in $\arg\lambda \in [-(\pi - \sigma)/2, (\pi - \sigma)/2]$. Comparing equations (9) and (16) and using the uniqueness theorem in $B_0^0(R_n)$ for the solution of equation (9), we obtain that the function $\widetilde{w}(x, \lambda)$ is analytically continued into the domain $|\arg\lambda| < \pi$ by the equality $\widetilde{w}(x, \lambda) = \tilde{z}(x, \sqrt{\lambda})$, and $\widetilde{w}(x, \lambda) \to 0$ as $|\lambda| \to \infty$ uniformly in $\arg\lambda \in [-\pi + \sigma, \pi - \sigma]$. Since $w(x, t) = u_1(x, t)$ for $t \geq \delta$, from (10) we have

$$\widetilde{w}(x, \lambda) = -\mathcal{L}(\lambda)\widetilde{w}(x, \lambda) - \mathcal{L}(\lambda)u(x, \delta) - u(x, \delta) + \lambda\tilde{v}(x, \lambda).$$

From (11) and (12) it follows that

$$\|\widetilde{w}(x, \lambda)\|_{B_0^0(R_n)} \leq \text{const} \quad \text{for } |\lambda| < \varepsilon,\ |\arg\lambda| \leq \pi - \sigma.$$

Replacing, in the inverse Laplace transform for $\widetilde{w}(x, \lambda)$, the contour of integration $\operatorname{Re}\lambda = \beta > 0$ by the contour $|\arg\lambda| = 3\pi/4$ (we assume that $\sigma < \pi/4$), we easily obtain the estimate (13).

Lemma 3. *If, for a continuously differentiable function $f(x, t)$, the conditions*

$$\frac{1}{t}\int_0^t f(x, \tau)\, d\tau \to 0 \quad \text{as } t \to \infty \tag{17}$$

hold uniformly in $x \in R_n$, and

$$|\partial f(x, t)/\partial t| \leq C/t \quad \text{for } t \geq t_0 \tag{18}$$

with some constant $C > 0$, then

$$f(x, t) \to 0 \qquad \text{as } t \to \infty$$

uniformly in $x \in R_n$.

This lemma is a consequence of Theorem 33 of the book ([4]) (p. 388). Let us take as the function $f(x, t)$ the function $u(x, t) - v(x, t)$. By Lemma 1, relation (17) holds for this function, and by Lemma 2 relation (18) holds, since the inequality $|\partial v(x, t)/\partial t| \leq C/t$ for $t > 0$ is obvious. Therefore it follows from Lemma 3 that

$$u(x,t) - v(x,t) \to 0 \qquad \text{as } t \to \infty$$

uniformly in $x \in R_n$. After this, the theorem formulated above follows from known results ([1,2,5,6]) concerning the solution $v(x,t)$ of the Cauchy problem for the heat equation, since from condition (5) it follows that, for relation (7) to hold, it is necessary and sufficient that the relation

$$\frac{n}{\omega_n R^n} \int_{|x-y| \le R} p(y)\varphi(y)\, dy \to A$$

hold.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR
Moscow

Received
18 II 1970

REFERENCES

[1] V. P. Mikhailov, DAN, **190**, No. 1 (1970).
[2] N. Wiener, *The Fourier Integral and Certain of Its Applications*, Moscow, 1963.
[3] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Leningrad, 1950.
[4] G. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, IL, 1948.
[5] B. D. Repnikov, S. D. Eidelman, DAN, **167**, No. 2 (1966).
[6] Yu. N. Drozhzhinov, Izv. Akad. Nauk SSSR, Ser. Mat., **33**, No. 2 (1969).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.