Academician A. D. ALEKSANDROV

ft(M)=t'f(M),

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Abstract

Full Text

MATHEMATICS

Academician A. D. ALEKSANDROV

MAPPINGS OF FAMILIES OF SETS

1. Let $A_n, A_{m'}$ be affine spaces ($n \geq 1$), let M be a bounded set in A_n, and let f be a one-to-one mapping of A_n onto $A_{m'}$ having the following property. The image of every set $t(M)$, obtained from M by some parallel translation, is $t'(M')$, obtained by translating the set $M' = f(M)$, and conversely: every $t'(M')$ is the image of some $t(M)$. In short, the equality holds

$$ft(M) = t'f(M), \tag{1}$$

where to each t there corresponds t', and conversely, so that f realizes a mapping of the family $\{t(M)\}$ onto $\{t'(M')\}$. The question consists in investigating mappings f of this kind. Obviously, every affine mapping of A_n onto $A_{m'}$ will be a mapping f for any M; what other mappings f exist for a given M?

2. To formulate our results, we introduce some definitions. Let P be any $(n-1)$-dimensional plane in A_n, parallel to some given P_0, and let l be a vector not parallel to P.

We shall call a **shift of the planes** P, **preserving** l, such a homeomorphism d of the space A_n onto itself that: 1) d takes every directed segment equal to l into a segment equal to l; 2) on every plane P, d is its parallel translation. This is equivalent to the following. If in A_n we introduce coordinates $x_1, \dots, x_n$ so that the axis x_1 is parallel to l, and the axes $x_2, \dots, x_n$ are parallel to P, then the mapping d_{Pl} is represented by the formulas:

$$x_1' = \varphi(x_1), \quad x_2' = x_2 + a_2, \dots, \; x_n' = x_n + a_n, \tag{2}$$

where the a_i are constants, and φ is such a homeomorphism of the axis x_1 onto itself that $\varphi(x_1 + |l|) = \varphi(x_1) + |l|$.

Obviously, every shift $d = d_{Pl}$ takes any cylinder C with generator l and base plane P into the same kind of cylinder, i.e., there exists a translation t such that $d(C) = t(C)$. We shall call a **quasicylinder** any (nonempty) bounded set Q possessing the same property. In this case P is called the base plane of the quasicylinder Q, and l its generator, under the additional condition that for no vector $kl \neq \pm l$ (k an integer) does Q possess the same property. (That is, shifts $d = d_{Pkl}$ are found such that $d(Q) \neq t(Q)$. Otherwise we could regard as the "generator," for example, half the generator of an ordinary cylinder.)

The case $l = 0$ is not excluded, when the quasicylinder Q is a plane set: $Q \subset P$, and the shift d_{Pl} is any homeomorphism of A_n onto itself that is a translation on every plane P. At the same time it is not difficult to see that a set N will be a quasicylinder with base plane P and generator $l \neq 0$ if and only if it admits the following representation. There are planes $P_1, \dots, P_p$ parallel to P, the distances between which in the direction l have common measure $|l|$, and N is composed of sets lying in the planes P_i, and of open segments parallel to l with endpoints on the planes P_i. In this case it is not necessary that such segments occur between every pair P_i, P_j—there may be none at all, just as it is not necessary that on every P_i there be points of N; but no plane P_i can-

be excluded without the indicated representation of the set N_φ, with l replaced by $kl \neq \pm l$, ceasing to be possible.

One and the same quasicylinder may have several generating and base planes, as, for example, a parallelepiped: it has n generating and n base planes.

We say that a quasicylinder Q is degenerate if there exists a base plane P such that Q is contained in a finite number of planes parallel to P; otherwise Q is nondegenerate. Note that if a nondegenerate quasicylinder has two generators l_1, l_2 and base planes P_1, P_2, then, as can be shown, $l_1 \parallel P_2$ and $l_2 \parallel P_1$. Hence it follows, in particular, that a nondegenerate quasicylinder can have at most n generators and base planes.

3. From the definitions of the displacement d_{Pl} and of a quasicylinder it follows immediately that if the set $M \subset A_n$ is a quasicylinder with base plane P and generator l, then for every displacement $d = d_{Pl}$ and every translation t there exists a translation $\bar{t}$ such that

$$dt(M) = \bar{t}(M). \tag{3}$$

Therefore, if M has generators $l_1, \dots, l_k$ and base planes $P_1, \dots, P_k$, then for any $d_i = d_{P_i l_i}$ and any translation t there also exists a translation $\bar{t}$ such that

$$d_1 \dots d_k t(M) = \bar{t}(M). \tag{4}$$

This is nothing other than formula (1) with $f = d_1 \dots d_k$, $f(M) = M$, $t' = \bar{t}$. Thus the product of the displacements d_i is a mapping f of the space A_n onto itself, as defined in item 1. If, in addition, we perform any affine mapping f_0 of A_n onto A'_n, then we obtain a mapping f of A_n onto A'_n:

$$f = f_0 d_1 \dots d_k. \tag{5}$$

At the same time the following holds.

Theorem 1. *Let the mapping f, defined in item 1, be continuous (so that $m = n$). Then, if the set M is not a quasicylinder, f is affine. If M is a* *nondegenerate quasicylinder having exactly k generators l_i and base planes P_i, then f is representable in the form (5), where f_0 is an affine mapping of A_n onto A'_n and d_i are the displacements $d_{P_i l_i}$. Moreover, one may take $f_0(M) = f(M)$, so that also in this case the set $M' = f(M)$ is an affine image of M.*

Finally, if M is a degenerate quasicylinder, then continuous mappings f may be quite arbitrary.

However, they are easy to characterize, relying on the preceding assertions of the theorem. For example, if for a base plane P there is a $P_1 \parallel P$ such that $M \cap P_1$ is not a quasicylinder in P_1 (and is nonempty), then f is affine on every plane $P' \parallel P$.

Note that for a nondegenerate quasicylinder the displacements d_i commute, so that their order in (5) is immaterial, and repeated application of them leads to the same formula (5). This follows from the fact that, for a nondegenerate quasicylinder, when $i \neq j$, $l_i \parallel P_j$ and $l_j \parallel P_i$.

4. Theorem 1 reduces the question of the nature of the mappings f to the question of the conditions for their continuity. To formulate such, possibly more general, conditions, we introduce the following construction.

Fix a point $O \in A_n$ and, putting $M = M_0$, define by induction the sets M_i: M_i is the union of all $t(M_{i-1})$ that contain O:

$$M_i = \bigcup t(M_{i-1}); \qquad O \in t(M_{i-1}), \quad M_0 = M. \tag{6}$$

Theorem 2. *If $M' = f(M)$ is bounded, then f is continuous under one of the following conditions: (I) among the sets M_i there are open ones; (II) among the M_i there are closed sets with interior points and f^{-1} is bounded.*

On the other hand, for any given M there exist discontinuous mappings f of the space A_n onto any A_m.

Let us note that condition (I) of Theorem 2 is certainly satisfied if M itself is open, or even $M = G \setminus N$, where G is open and $\overline{N} \subset G$. Likewise, condition (II) on M_i is satisfied if M itself is closed with interior points, or $M = F \setminus N$, where F is closed with interior points and N is contained inside F. Therefore, in particular, f is continuous if M is the boundary of a domain and M' is bounded, and f^{-1} is bounded. At the same time, if M is closed but none of the sets M_i has interior points, then a discontinuous mapping f is certainly possible, even if M' is bounded and f^{-1} is bounded.

If among the M_i there are closed sets with interior points, then the requirement that f^{-1} be bounded is probably superfluous, but we can prove this only in special cases, for example when some M_i is a polyhedron or a strictly convex body. In general this requirement can be replaced, for example, by one of the two following requirements: (a) the $f(M_i)$ are measurable; (b) the interior measure of one of the $f(M_i)$ is positive.

Theorem 2 can also be supplemented by conditions ensuring the continuity of f also when none of the sets M_i is either open or closed. However, the question of necessary and sufficient conditions on M under which f must be continuous (for bounded M') remains open.

5. The construction (6) of the sets M_i can be represented as follows.

Take the point O as the origin of the vectors. Then, as is easy to see, the set M_i is formed by the endpoints of all vectors $y - x$, where the endpoints of the vectors y and x independently run through the whole set M_{i-1}. Thus, M_i is the set of endpoints of the vectors

$$\sum_{1}^{i}(y_j - x_j),$$

where the endpoints y_j and x_j run through the set M.

The construction therefore consists in the successive construction of the additive group of vectors generated by the set M as a set of vectors. Hence the condition that the sets M_i cover all of A_n,

$$\bigcup_{i=0}^{\infty} M_i = A_n, \tag{7}$$

is equivalent to saying that M is a generating set of the entire group A_n.

If among the M_i there are sets having interior points, then (7) is obviously satisfied. Let us also note that if some M_k is closed or open, then the same is true for all M_i, $i > k$. Therefore, in particular, if M_k is closed and (7) holds, then the whole space A_n is represented as the sum of a countable number of closed sets $M_k \subset M_{k+1} \subset \cdots$. Then, by known theorems of dimension theory (see, for example, ([1])), among the M_{k+i} there is a set with interior points. Thus, the condition of Theorem 2 that among the M_i there is a closed set with interior points is equivalent to saying that among them there are closed sets and that (7) holds.

If (7) is not satisfied, then the mapping f certainly may fail to be continuous, even if M' is bounded and f^{-1} is bounded. Indeed, if $\bigcup M_i = N \neq A_n$, then take in A_n a point O_1 not belonging to N. Construct around it the sets M_{i1} in the same way as the M_i are constructed around O, and form their sum N_1. In the sense of the group of vectors, N is a subgroup of the whole group A_n, and N_1 is its adjacent coset, so that $N \cap N_1 = \emptyset$. Interchanging N and N_1 by parallel translations, we obtain a discontinuous mapping f of the space A_n onto itself.

It seems probable that, conversely, for the continuity of f it is sufficient that (7) be satisfied, provided only that M' is bounded.

6. We give one consequence of our results. Let a metric $\rho(XY)$ be given in the space A_n, satisfying the condition of invariance under parallel translation. The sets

$$\{Y : \rho(XY) \leq 1\}, \qquad \{Y : \rho(XY) < 1\}, \qquad \{Y : \rho(XY) = 1\}$$

will be, respectively, the unit "ball," the open ball, and the "sphere" with center X; a translation of the center $X \to X_1$ induces the same translation of each of them.

It is known ([2],[3]) that if in the spaces A_n, A'_n there are translation-invariant metrics ρ, ρ' and there is an isometric mapping g of A_n onto A'_n, then g is affine. From our Theorems 1 and 2, however, a much stronger result follows immediately.

Theorem 3. *Let translation-invariant metrics be given in the spaces A_n, A'_n, and suppose that the unit ball S in A_n is not a quasicylinder. Let g be a one-to-one mapping of A_n onto A'_n under which either unit balls, or open unit balls, or unit spheres in A_n are mapped onto the corresponding sets in A'_n, and conversely. Then g is affine.*

If, however, S is a quasicylinder, then g need not be affine, and the definitions are as in Theorem 1 (so that, in particular, the unit ball S' in A'_n is nevertheless an affine image of S). But for g to be affine it is sufficient, for example, that for each generator l_i of the ball S there be an irrational α_i such that for every pair of points X, Y with $\overline{XY} = \alpha_i l_i$ one has $\rho(XY) = \rho'(X'Y')$, where $X' = g(X)$, $Y' = g(Y)$ and ρ, ρ' are metrics in A_n, A'_n.

In view of the generality of Theorems 1 and 2, the symmetry of the metric, i.e. the condition $\rho(XY) = \rho(YX)$, is not at all essential.

7. The proof of Theorems 1 and 2, though elementary, is somewhat too complicated for us to outline it here; only the proof of the first part of Theorem 2 for open M turns out to be simple.

The construction (6) of the sets M_i, $i \geq 1$, plays the main role. First, it is easy to see that the point O will be their center of symmetry. Further, from the definition of the mapping f it follows directly that the sets $M'_i = f(M_i)$ are constructed from M' in exactly the same way about the point $O' = f(O)$. In general, if M_{iX} is constructed about the point X, then $M'_{iX'} = f(M_{iX})$ is constructed in the same way about $X' = f(X)$. At the same time M_{iX} is obtained from M_i by the translation $t : O \to X$, and $M'_{iX'}$ from M'_i by the translation $t' : O' \to X'$. Therefore for the sets M_i the same formula (1) holds:

$$ft(M_i) = t'f(M_i)$$

with the important additional fact that the center $t(M_i)$ is mapped to the center $t'f(M_i)$. In view of this we may, instead of the given M, consider any of the sets

M_i; and they are "better arranged." This is the starting point of our arguments.

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REFERENCES

1. V. **Gurevich**, G. **Volman**, *Dimension Theory*, Moscow, 1948, Theorems III 2 and IV 3.

2. M. M. **Day**, *Normed Linear Spaces*, Moscow, 1961, Ch. VII, § 1.

3. Z. **Chavzyński**, *Studia Math.*, **13**, 94 (1953).

Note: Figure translations are in progress. See original paper for figures.