DEFINITION OF COMPLEX MARKOV CHAINS AND DERIVATION OF THE SCHRÖDINGER EQUATION

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Abstract

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DEFINITION OF COMPLEX MARKOV CHAINS AND DERIVATION OF THE SCHRÖDINGER EQUATION

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1. Let $\Omega = \{A_1, A_2, \dots, A_n\}$ be a finite set with the σ-algebra $\mathfrak{A}$ of all its subsets, on which a probability $P(A)$, $A \in \mathfrak{A}$, is defined. We shall call the elements of the algebra $\mathfrak{A}$ true events, and the elements $A_1, A_2, \dots, A_n$ elementary true events. Let A_i itself be subdivided into k elements

$$A_i = \{B_1^i, B_2^i, \dots, B_k^i\},$$

which we shall call elementary virtual events. Their totality (for $i = 1, 2, \dots, n$) will be called the set of elementary events, and the elements of the σ-algebra $\mathfrak{B}^i$ of subsets of A_i will be called virtual events.

Let $Z(B)$ be a certain complex function on the σ-algebra $\mathfrak{B}$ generated by the set of nk elementary virtual events, additive on each of the sets A_i and such that

$$|Z(A)|^2 = P(A), \qquad A \in \mathfrak{A}. \tag{1}$$

We shall call the elements of the σ-algebra $\mathfrak{B}$ simply events, and the function $Z(B)$, $B \in \mathfrak{B}$, the amplitude of the event B.

Definition 1. The **conditional amplitude** $Z(B_1/B_2)$ of the event B_1 upon the occurrence of the event B_2 is defined as the ratio

$$Z(B_1/B_2) = \frac{Z(B_1 \cdot B_2)}{Z(B_2)}.$$

2. We shall consider a finite system $\mathcal{K}$ of states $a_1, \dots, a_k$, which replace one another at finite time intervals $T_0, T_1, \dots, \dots, T_N$. As elementary true events A_{ij} we take all possible pairs of points $\{(T_0, a_i), (T_N, a_j)\}^*$ in the plane T, a. We subdivide this event into elementary virtual events that are sequences of N points

$$\{(T_0, a_i), (T_1, a_{i_1}), \dots, (T_{N-1}, a_{i_{N-1}}), (T_N, a_j)\},$$

in which the initial and final points are fixed. In other words, the elementary virtual events are all possible "paths" of the system $\mathcal{K}$ from the point (T_0, a_i) to the point (T_N, a_j). The set of all paths joining all initial and final points is the set of elementary events.

Definition 2. Suppose that on the σ-algebra of paths an amplitude $Z(B)$ is given. We shall say that it **defines a complex Markov chain** (a **KM-chain**) if the conditional amplitude of the fact that the system is in the state** a_ν at the time T_{i+1}, under the condition that at the moments $T_i, T_{i-1}, \dots, T_0$ it was respectively in the states $a_\mu, a_{\mu_1}, \dots, a_{\mu_{i-1}}$, is equal to the conditional amplitude of the fact that the system is in the state a_ν at the time T_{i+1}, under the condition that at the time T_i it was in the state a_μ. We shall call this amplitude the **transition amplitude** from a_μ to a_ν in one step and denote it by $z_{\mu\nu}^{i,i+1}$.

Definition 3. A KM-chain is called **homogeneous** if its one-step transition amplitudes do not depend on T_i and N.

Let us put $N = 1$ in a homogeneous KM-chain. In this case there are no virtual events. Therefore we shall regard such a situation as purely classical, and shall obtain the transition probabilities $P_{\mu\nu}^{0,1} = |z_{\mu\nu}^{0,1}|^2$ from considerations of classical probability theory. The transition probabilities for a homogeneous KM-chain with $N = 1$ will be called the **classical transition probabilities** of the KM-chain (for arbitrary N).

Let us pass to continuous KM-processes. Let the system $\mathcal{K}$ consist of the points $x_\nu = \varepsilon\nu$, $|x_\nu| \le l(\varepsilon)$, where $l(\varepsilon) \to \infty$ as $\varepsilon \to 0$, and suppose, moreover, that $T_{i+1} - T_i \to 0$ as $\varepsilon \to 0$, while $N \to \infty$ in such a way that $T_N \to t$. We shall weaken relation (1), putting

$$|Z(A)|^2 = P(A) + o(\varepsilon), \qquad A \in \mathfrak{A}. \tag{2}$$

In this case (with the other conditions preserved) we shall say that the amplitude $Z(B)$ **defines** a homogeneous ε-KM-chain (cf. (1)).

We shall say that a homogeneous ε-KM-chain **defines a continuous** KM**-process** if there exists a function $G(x, \xi, t)$ such that: 1) $\left|G(\mu\varepsilon, \nu\varepsilon, T_i) - \varepsilon^{-1} Z((T_i, x_\nu)/(T_0, x_\mu))\right| \to 0$ as $\varepsilon \to 0$, uniformly in μ, ν, i; 2)

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* This is connected with the physical fact that we can perform measurements only at the initial and final moments.

** The event (T_i, a_ν), consisting in the fact that the system at the time T_i is in the state a_ν, is the subset of all paths passing through the point (T_i, a_ν). The true event $A_{\mu\nu}$ is equal to the intersection of the event (T_0, a_μ) and the event (T_N, a_ν).

the convolutions, in the second argument, of the first derivatives of $G(x, \xi, t)$ at $t = 0$ with functions of the class S (the fundamental functions) belong to S; 3) at those points x at which $G(x, \xi, t)$ is less than some constant, it is infinitely differentiable. The function $G(x, \xi, t)$ will be called the **Green' s function** of the limiting KM-process.

Now consider nonrelativistic particles in an electromagnetic field, emitted from the point ξ with an equiprobable distribution of velocities.

The velocity of a particle which, in the time t, has arrived at the point x_i, is equal to $v_i = (x_i - \xi)/t + O(1)$ as $t \to 0$. The probability of arriving in the interval $[x_1, x_2]$ is, obviously, proportional to $v_2 - v_1 = (x_2 - x_1)/t + O(1)$. Therefore, for the ε-KM-chain corresponding to the motion of a nonrelativistic particle in an electromagnetic field, according to what has been said above, the relation

$$\sum_{j_1 \leq j \leq j_2} P_{ij}^{0,1} = (x_{j_2} - x_{j_1})/(T_1 - T_0) + O(1), \tag{3}$$

must hold, where $\beta P_{ij}^{0,1}$ are the classical transition probabilities of the KM-chain, β is a normalizing constant.

Theorem 1. In order that $G(x, \xi, t)$ be the Green' s function of a KM-process that is limiting for some ε-KM-chain whose classical transition probabilities satisfy condition* (3), it is necessary and sufficient that it satisfy the equation (Schrödinger)

$$i\, \partial G/\partial t = \{[i\alpha\, \partial/\partial x + A(x)]^2 + V(x)\}G,$$

where $A(x)$, $V(x)$ are certain smooth functions, α is a constant.

The proof of sufficiency follows easily from the asymptotics, as $t \to 0$, of the Green' s function of the Schrödinger equation. We outline the proof of necessity. It is based on a number of lemmas.

Lemma 1. The conditional amplitude $Z((T_{i+j}, a_\nu)/(T_i, a_\mu))$ of a KM-chain is equal to the matrix element $a_{\mu\nu}$ of the product of matrices

$$\mathrm{Mat}\, a_{\mu\nu} = \mathrm{Mat}\, z_{\mu\nu}^{i,i+1}\, \mathrm{Mat}\, z_{\mu\nu}^{i+1,i+2} \ldots \mathrm{Mat}\, z_{\mu\nu}^{i+j-1,i+j}.$$

The proof is analogous to the proof of the corresponding assertion for Markov chains (2). It follows from it that

Lemma 2. The operator

$$U_t \psi(x) = \int G(x, \xi, t) \psi(\xi)\, d\xi,$$

where $G(x, \xi, t)$ –

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* That is, for a nonrelativistic particle in an electromagnetic field.

function, the Green function of the limiting CM-process, is a unitary one-parameter group.

Lemma 3. For the Green function of the limiting CM-process satisfying the condition of the theorem, the following asymptotic relation holds:

$$|G(x, \xi, t)|^2 = \mathrm{const}/t + O(1) \tag{4}$$

as $t \to 0$.

Let $\hat{H}$ be the generating self-adjoint operator of the group U_t. The Green function $G(x, \xi, t)$ obviously satisfies the equation

$$i\,\partial G/\partial t = \hat{H} G, \qquad G|_{t=0} = \delta(x - \xi). \tag{5}$$

From relation (4) one may conclude that $G(x, \xi, t)$ has a singularity only at $t = 0$. Consequently, the approximate solution of problem (5), obtained up to smooth functions, coincides with the asymptotics of the solution as $t \to 0$ (cf. (3)). The operator $\hat{H}$ can be represented as the pseudodifferential operator

$$\hat{H}\psi(x) = i \lim_{t\to 0} \int (\partial G/\partial t)\psi(\xi)\, d\xi = \int H(p, x)\psi(\xi) \exp[ip(x - \xi)]\, d\xi, \quad \psi(x) \in S \tag{6}$$

with a certain complete symbol $H(p, x)$, which, by virtue of property 2) for $G(x, \xi, t)$, turns out to be sufficiently smooth and to grow more slowly than any exponential.

The construction of the solution of problem (5), where $\hat{H}$ is a pseudodifferential operator, up to smooth functions (for small t) is carried out by the method of the work (^{5}). It turns out that the following is valid.

Lemma 4. The asymptotics as $t \to 0$ of the solution of problem (5) satisfying condition (4) has the form

$$\left|\partial^2 S/\partial x\,\partial\xi\right|^{-1/2} \{e^{iS(x,\xi,t)} + O(t)\},$$

where $S(x, \xi, t)$ satisfies the Hamilton–Jacobi equation

$$\partial S/\partial t + H(\partial S/\partial x, x) = 0.$$

Hence, and from (4), it is concluded that

$$S(x,\xi,t)=a(x-\xi)^2/t+O(1).$$

Therefore

$$\int G(x,\xi,t)(x-\xi)^3F(\xi)\,d\xi=o(t) \tag{7}$$

for any finite $F(\xi)$.

From equality (6) it follows that the limits* exist

$$\lim_{t\to 0}t^{-1}\left[\int G(x,\xi,t)e(\xi)\,d\xi-e(x)\right]=V_N(x),$$

$$\lim_{t\to 0}t^{-1}\int G(x,\xi,t)(x-\xi)e(\xi)\,d\xi=A_N(x), \tag{8}$$

$$\lim_{t\to 0}t^{-1}\int G(x,\xi,t)(x-\xi)^2e(\xi)\,d\xi=2B_N(x),$$

where $e(\xi)$ is a finite function equal to 1 for $\xi\in(-N,N)$.

Let $\psi_0(x)$ be finite, $\operatorname{supp}\psi_0\in(-N,N)$. Since

$$\psi_0(\xi)=\psi_0(x)e(\xi)+(x-\xi)\psi_0'(x)e(\xi)+$$

$$+\frac{1}{2}(x-\xi)^2\psi_0''(x)e(\xi)+(x-\xi)^3F(x,\xi)e(\xi),$$

then, by virtue of (7) and (8),

$$\int G(x,\xi,t)\psi_0(\xi)\,d\xi-\psi_0(x)=$$

$$=t\,[V_N(x)\psi_0(x)+A_N(x)\psi_0'(x)+B_N(x)\psi_0''(x)]+o(t).$$

* Cf. A. N. Kolmogorov' s derivation of the equation for a limiting Markov process.

Consequently,

$$\hat{H}\psi_0=B_N\psi_0''+A_N\psi_0'+V_N\psi_0. \tag{9}$$

Hence, again taking into account (4) and Lemma 4, and then carrying out the closure procedure ($N \to \infty$), we arrive at the assertion of the theorem, since, by virtue of (9), for $N_1 < N_2$ and $x \in (-N_1, N_1)$ the equalities $A_{N_1}(x) = A_N(x)$, $B_{N_1}(x) = B_N(x)$, $V_{N_1}(x) = V_N(x)$ hold.

The generalization to the inhomogeneous (time-dependent) multidimensional case can be carried out without substantial changes.

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Note: Figure translations are in progress. See original paper for figures.

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