CUBATURE FORMULAS IN \(L_2^{(m)}(\Omega)\)

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Abstract

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CUBATURE FORMULAS IN $L_2^{(m)}(\Omega)$

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In ([1−5]) Sobolev investigated formulas of approximate integration in the classes $L_2^{(m)}(E_n)$, $L_2^{(m)}(\Omega)$. The case $L_2^{(m)}(E_n)$ was studied most fully. In particular, cubature formulas with a regular boundary layer and with error functionals were determined ([2,4])

$$l^h \in L_2^{(m)*}(E_n), \quad 2m > n,$$

$$(l^h, f) = \int_\Omega pf\,dx - \sum_{hH\gamma\in\Omega} c_\gamma^h f(hH\gamma): \int |p|^2\,dx = \|p\|^2_{L_2(\Omega)} < \infty, \qquad (1)$$

where $p(x) \equiv 1$, and such that, as $h \to 0$ and for given p,

$$\|l^h\|^2_{B^*} - B_{n,m}(H)\|p\|^2_{L_2(\Omega)} h^{2m} = o(h^{2m}): \quad B_{n,m}(H) = (2\pi)^{-2m} \sum_\beta{}' |\beta H^{-1}|^{-2m}, \qquad (2)$$

if $B = L_2^{(m)}(E_n)$. It was shown ([2,5]) (see also ([7])) that for any $\{l^h\}$ of the form (1) and $B = L_2^{(m)}(E_n)$, the left-hand side of (2) is greater than some $\alpha(h)$, $\alpha(h) = O(h^{2m+1})$ as $h \to 0$, i.e., that families of formulas with a regular boundary layer are asymptotically optimal on lattices in $L_2^{(m)}(E_n)$, or, in other words, the asymptotically optimal families of error functionals of these formulas. In ([6]) the indicated results were generalized to the case $p \in L_2(\Omega)$, and it was shown that for such weights the functionals l^h, forming an asymptotically optimal family, satisfy (2) with $B = L_2^{(m)}(E_n)$. The methods of proving the assertions in ([1−6]) were based on the theory of the polyharmonic equation $\Delta^m u = 0$, which is the Euler equation for the functionals (1) in $L_2^{(m)}(\Omega)$, $L_2^{(m)}(E_n)$. The case of the spaces $L_2^{(m)}(E_n)$ turned out to be simpler than the case $L_2^{(m)}(\Omega)$, since in the study of formulas of approximate integration in $L_2^{(m)}(E_n)$ no boundary-value problems arise for the polyharmonic equation ([1,3]). In the present paper the results from ([1–6]) concerning asymptotically optimal families of cubature formulas in $L_2^{(m)}(E_n)$ are generalized to $L_2^{(m)}(\Omega)$ by means of methods of the theory of Hilbert spaces.

Notation: Ω is a domain in the n-dimensional Euclidean space E_n with a sufficiently "good" boundary (for example, piecewise smooth); γ are integer vectors from E_n; H is an $n \times n$ matrix, $|H| = 1$; $L_2 = L_2(\Omega)$; h is a positive parameter; m is an integer greater than $n/2$; $L_2^{(m)}(E_n)$, $L_2^{(m)}(\Omega)$ are known Banach spaces of classes of functions that may be regarded as Hilbert spaces ([1]), if the scalar product in them is defined as follows: let $F, G \in L_2^{(m)}(M)$ (M is one of the sets E_n, Ω); f, g are functions, $f \in F$, $g \in G$; then

$$(F, G)_{L_2^{(m)}(M)} = \int_M \left(\sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \frac{\partial^m f}{\partial x_{j_1} \cdots \partial x_{j_m}} \frac{\partial^m g}{\partial x_{j_1} \cdots \partial x_{j_m}} \right) dx.$$

$\dot{L}_2^{(m)}(\Omega, E_n)$, $\dot{L}_2^{(m)}(E_n \setminus \Omega, E_n)$ are subspaces of $L_2^{(m)}(E_n)$, defined as follows: $F \in \dot{L}_2^{(m)}(M, E_n)$ (M is one of the sets $\Omega, E_n \setminus \Omega$) if and only if there is a function f in F that is zero in $E_n \setminus M$; Π is the operator assigning to $F \in L_2^{(m)}(E_n)$ the class $\Pi F \in L_2^{(m)}(\Omega)$ consisting of the traces on Ω of functions from F; $\dot{L}_2^{(m)}(\Omega)$ is the subspace of $L_2^{(m)}(\Omega)$ consisting of images of elements of $\dot{L}_2^{(m)}(\Omega, E_n)$ under the mapping Π; $U_2^{(m)}(\Omega) = L_2^{(m)}(E_n) \ominus \dot{L}_2^{(m)}(E_n \setminus \Omega, E_n)$; R is the restriction of Π to $U_2^{(m)}(\Omega)$;

$$H_2^{(m)}(\Omega) = L_2^{(m)}(\Omega) \ominus \dot{L}_2^{(m)}(\Omega) : \quad H_2^{(m)}(\Omega, E_n) = L_2^{(m)}(E_n) \ominus \dot{L}_2^{(m)}(\Omega, E_n). \quad (3)$$

Theorem 1. If $p \in L_2$, and $\{l^h\}$ is a family of functionals (1) such that, as $h \to 0$, (2) is fulfilled with $B = L_2(E_n)$, then (2) is also fulfilled for $B = L_2^{(m)}(\Omega)$.

Lemma 1. If p, $\{l^h\}$ satisfy the conditions of Theorem 1, then (2) is fulfilled for $B = \dot{L}_2^{(m)}(\Omega, E_n)$.

We outline the proof of the lemma. Since

$$\|l^h\|_{L_2^{(m)*}(E_n)} \geqslant \|l^h\|_{\dot{L}_2^{(m)}(\Omega, E_n)},$$

it suffices to prove the existence of $\{V_h\} \subset \dot{L}_2^{(m)}(\Omega, E_n)$ such that, as $h \to 0$,

$$\int_\Omega p V_h \, dx = \|p\|_{L_2} \sqrt{B_{n,m}(H)} \, h^m \|V_h\|_{L_2^{(m)}(E_n)} + o(h^m) : \quad V_h(hH\gamma) = 0. \quad (4)$$

For $p = 1$, the existence of the indicated $\{V_h\}$ was proved in (7). It is not difficult to construct the necessary $\{V_h\}$ if p is piecewise constant in Ω. Let now $g \in L_2$ be arbitrary, p piecewise constant, and let $\{V_h\} \subset \dot{L}_2^{(m)}(\Omega, E_n)$ satisfy

(4); let $\{I^h\}$ be a uniformly distributed family of interpolation operators (for the definition of such operators see (6)), interpolating at the points $h\gamma$. Then

$$I = \int_\Omega g V_h \, dx = \int_\Omega (g-p)(V_h - I_h^{hV})\, dx + \int_\Omega p V_h \, dx = I_1 + I_2.$$

From Theorem 1 in (6) it follows that if p is close to g in the metric of L_2, then I_1 is small and $I \cong I_2$. Using the approximate equality obtained and taking into account the density of piecewise constant functions in L_2, one can prove the lemma in the general case.

We proceed directly to the proof of the theorem. From (3) it follows that

$$\begin{aligned} \|l^h\|^2_{L_2^{(m)*}(\Omega)} &= \|l^h\|^2_{\dot{L}_2^{(m)*}(\Omega)} + \|l^h\|^2_{H_2^{(m)*}(\Omega)} : \quad \|l^h\|^2_{L_2^{(m)*}(E_n)} \\ &= \|l^h\|^2_{\dot{L}_2^{(m)*}(\Omega,E_n)} + \|l^h\|^2_{H_2^{(m)*}(\Omega,E_n)}. \end{aligned} \tag{5}$$

Lemma 1 and (5) give

$$\|l^h\|^2_{L_2^{(m)*}(\Omega)} = B_{h,m}(H)\|p\|^2_{L_2} h^{2m} + o(h^{2m}) + \|l^h\|^2_{H_2^{(m)*}(\Omega)}; \quad \|l^h\|^2_{H_2^{(m)*}(\Omega,E_n)} = o(h^{2m}). \tag{6}$$

If we show that

$$\|l^h\|_{H_2^{(m)*}(\Omega)} \leqslant K \|l^h\|_{H_2^{(m)*}(\Omega,E_n)}, \tag{7}$$

where K is independent of h, then Theorem 1 will follow from (6), (7).

All elements of $L_2^{(m)}(\Omega)$ can be extended to elements of $L_2^{(m)}(E_n)$ (see, for example, ([1,8])) and $U_2^{(m)}(\Omega)$, whence, as is not difficult to see, there follows the existence for R of a single inverse operator R^{-1}, defined on all of $L_2^{(m)}(\Omega)$. From Banach' s theorem on inverse operators it follows that R^{-1} is bounded as an operator acting from $L_2^{(m)}(\Omega)$ into $U_2^{(m)}(\Omega)$. The boundedness of R^{-1} and the fact that all possible extensions of elements of $H_2^{(m)}(\Omega)$ to elements of $U_2^{(m)}(\Omega)$ belong to $H_2^{(m)}(\Omega, E_n)$ give (7) and the assertion of Theorem 1.

Corollary. Let $\{l^h\}$ be asymptotically optimal in $L_2^{(m)}(E_n)$. Then $\{l^h\}$ satisfies the conditions of Theorem 1 (6) (for the case $p = 1$, see ([2,5])). From this theorem and the inequality

$$\|l^h\|_{L_2^{(m)*}(\Omega)} \geq \|l^h\|$$

it follows that $\{l^h\}$ is asymptotically optimal in $L_2^{(m)}(\Omega)$.

Remark. If $p = 1$ and $\{l^h\}$ is the collection of error functionals of formulas with a regular boundary layer, then the left-hand side of (2), for $B = L_2^{(m)}(\Omega)$, is $O(h^{2m+1})$, as $h \to 0$.

The proof of this assertion is analogous to the proof of Theorem 1.

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Note: Figure translations are in progress. See original paper for figures.