ASYMPTOTIC BEHAVIOR OF QUASIANALYTIC FUNCTIONS

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Abstract

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ASYMPTOTIC BEHAVIOR OF QUASIANALYTIC FUNCTIONS

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1°. Let $C\{M_n\}$ be a quasianalytic class of functions $f(x)$ defined on the entire real axis, $\|f\| = \sup_n M_n^{-1} \max_x |f^{(n)}(x)| < \infty$. Denote by $I_a(F)$ the subset of $C\{M_n\}$ consisting of functions $f(x)$ for which $|f(x)| = O(F(x-a))$. B. I. Korenblum ([1]) established the existence of such a function F that $I_a(F)$ is a nonempty closed subspace of $C\{M_n\}$ and $\bigcap_a I_a(F) = \emptyset$. This function determines the limiting rate of decrease in the class $*C\{M_n\}$.

In the present note it is shown that every function from the class $C\{M_n\}$ is representable as the sum of a function from $I_a(F)$ and a function holomorphic and belonging to the class $**\,A$ in a certain narrowing infinite strip $D_a(F)$. The strip $D_a(F)$ is characterized by the following property: if $\Psi(z)$ maps $D_a(F)$ conformally onto the right half-plane, then $\Psi(x) = O(-\ln F(x))$. This result makes it possible to establish an analogue of the well-known Ahlfors-Heins theorem ([4]) for quasianalytic functions and thereby to give a description of their asymptotic behavior along the real axis.

2°. Without loss of generality one may assume that the sequence $\{M_n\}$ is logarithmically convex, $\lim_{n\to\infty}(n^{-1}M_n)^{1/n} = \infty$, $M_0 = 1$ ([5]).

Simple computations show that, under these conditions, the function

$$\omega(x) = -\ln\left[x^4 \min_{n\geq 0} x^{-n} M_n\right], \quad x \geq 1,$$

has the following properties: $0 \leq \omega(x_1+x_2) \leq \omega(x_1)+\omega(x_2)$,

$$\frac{x\omega'(x)}{\omega(x)} \nearrow 1 \quad \text{as } x\to\infty, \qquad \int_1^\infty x^{-2}\omega(x)\,dx < \infty.$$

Put $\omega(\varphi_n) = n$,

$$\alpha(x) = \prod_{n=1}^{\infty}\left(1 + x^2\varphi_n^{-2}\right)^{1/2}.$$

The following lemma establishes the connection between functions of the class $C\{M_n\}$ and Fourier transforms of functions integrable with weight $\alpha(x)$.

Lemma 1. Let $f(x) \in C\{M_n\}$,

$$g(x) = \int_{-\infty}^{\infty} f(\tau)(1+\tau^2)^{-1} e^{ix\tau}\, d\tau.$$

* An earlier weaker result was obtained by Hirschman ([2]).

** A function $\varphi(z)$ belongs to the class A in the strip $D_a(F)$ if the image φ under a conformal mapping of $D_a(F)$ onto the right half-plane belongs to the class A (([3]), p. 289).

Then

$$\|g\|_\alpha = \int_{-\infty}^{\infty} |g(x)|\alpha(x)\, dx \leq K\|f\|,$$

where K does not depend on f.

Put

$$\xi(x) = \sum_{n=1}^{\infty} \frac{x^2}{\varphi_n(x^2+\varphi_n^2)}, \qquad \eta(\xi(x)) = x,$$

$$F(x) = \begin{cases} \exp\left[-\frac{1}{10}\exp\left(\frac{\pi}{2}\int_1^x \frac{\eta(t)}{\omega(\eta(t))}\, dt\right)\right], & \text{for } x \geq 2, \\ F(2), & \text{for } x < 2, \end{cases}$$

and, for real a, denote by $D_a(F)$ the domain in the complex plane $z = \sigma + i\tau$ defined by the inequalities

$$|\tau| \leq \frac{\omega(\eta(\sigma - a))}{\eta(\sigma - a)}$$

for $\sigma \geq a+1$, and $|\tau| \leq \omega(\eta(1))/\eta(1)$ for $\sigma < a+1$.

Lemma 2. If $\Psi(z)$ maps $D_1(F)$ conformally onto the right half-plane, then $\Psi(x) = O(-\ln F(x))$.

For functions integrable with weight $\alpha(x)$, the following holds.

Theorem 1. Let

$$\int_{-\infty}^{\infty} |g(x)|\alpha(x)\, dx < \infty, \qquad \tilde{g}(x) = \int_{-\infty}^{\infty} g(t)e^{ixt}\, dt.$$

Then for every $a \in (-\infty, \infty)$ there is a representation

$$\tilde{g}(x) = \tilde{g}_a^-(x) + \tilde{g}_a^+(x),$$

where $\tilde{g}_a^+(x)$ is the restriction to the real axis of an entire function $\tilde{g}_a^+(z)$ belonging to class A in the domain $D_a(F)$, $\tilde{g}_a^-(x) \in I_a(F)$, and

$$\sup_x |\tilde{g}_a^-(x)| \le c_g(a)\|g\|_\alpha, \qquad \lim_{a\to-\infty} c_g(a) = 0.$$

Remark 1. From the results of Warschawski ([6]) it follows that, if a function $\psi(z)$ belongs to class A in the strip $D_a(F)$ and $\psi(\sigma) \in I_{a-\varepsilon}(F)$, $\varepsilon > 0$, then $\psi(z) \equiv 0$. Therefore $\tilde{g}_a^+(x)$ is the principal part of the function $\tilde{g}(x)$.

Remark 2. The strip $D_a(F)$ in Theorem 1 cannot be substantially enlarged. Namely, it cannot be replaced by any domain containing every strip $D_a(F)$ for sufficiently large σ.

The main role in the proof of Theorem 1 is played by

Lemma 3. Let

$$\Phi(x) = \prod_{n=1}^{\infty} \left(1 - \frac{ix}{\varphi_n}\right)^{-1} e^{ix/\varphi_n}, \qquad \tilde{\Phi}(z) = \int_{-\infty}^{\infty} \Phi(x) e^{izx}\, dx, \qquad z = \sigma + i\tau.$$

Then:

a) $\tilde{\Phi}(z)$ is an entire function, bounded in every half-plane $\sigma \le \sigma_0$;

b) $\tilde{\Phi}(z)$ belongs to class A in the strip $D_0(F)$;

c) $\tilde{\Phi}(z) \in I_0(F)$.

Proof of Theorem 1. Since $\|g\|_\alpha < \infty$, we have $g(x) = \Phi(x) r(x)$, where $r(x) \in L^1(-\infty, \infty)$. Therefore

$$\tilde{g}(x) = \int_{-\infty}^{\infty} \tilde{\Phi}(x-t)\tilde{r}(t)\, dt = \int_{-\infty}^{a} \tilde{\Phi}(x-t)\tilde{r}(t)\, dt + \int_{a}^{\infty} \tilde{\Phi}(x-t)\tilde{r}(t)\, dt = \tilde{g}_a^-(x) + \tilde{g}_a^+(x).$$

Using assertion b) of Lemma 3 and the inclusion $D_p(F) \subset D_q(F)$ for $p < q$, we obtain that $\tilde{g}_a^+(x)$ is the restriction to the real axis of a function $\tilde{g}_a^+(z)$ belonging to class A in the strip $D_a(F)$. From assertion c) of Lem–

we and the inequality $2F(x) \ge \int_x^{\infty} F(t)\, dt$, we obtain that $\tilde{g}_a^-(x) \in I_a(F)$. The theorem is proved.

From Lemma 1 and Theorem 1 there follows the main

Theorem 2. Let $f(x) \in C\{M_n\}$. Then, for every a, $f(x)$ admits the representation

$$f(x) = f_a^+(x) + f_a^-(x),$$

where $f_a^+(x)$ is the contraction to the real axis of a function $f_a^+(z)$ belonging to the class A in the strip $D_a(F)$, $f_a^-(x) \in I_a(F)$, and

$$\sup_x |f_a^-(x)| \le c_f(a)\|f\|, \qquad c_f(a) \le 1, \qquad \lim_{a\to-\infty} c_f(a) = 0.$$

3°. Let us apply the results obtained to the study of the asymptotic behavior of functions of the class $C\{M_n\}$ as $|x| \to \infty$. To this end, we note that the asymptotic behavior of functions belonging to the class A in the domain $D_a(F)$ is known. Namely, from the Ahlfors-Heins theorem ([4]) and the results of Warschawski ([6]) it follows that, for a function $g(z) \ne 0$ belonging to the class A in $D_a(F)$, we have

$$\overset{*}{\lim_{\sigma\to\infty}} \frac{\ln|g(\sigma)|}{\ln F(\sigma - a)} = \gamma_g > -\infty,$$

where $\lim^*$ means that, in tending to infinity, σ does not take values from a certain set E, for which

$$\int_E \frac{\eta(x)}{\omega(\eta(x))}\,dx < \infty.$$

Combining this result with Theorem 2, we obtain the following theorem.

Theorem 3. Let $f(x) \in C\{M_n\}$, $f(x) \ne 0$. Then there exists an a such that*

$$\overset{*}{\lim_{x\to\infty}} \frac{\ln|f(x)|}{\ln F(x - a)} = 0.$$

An analogous result can be formulated for the case when $x \to -\infty$. Theorem 3 should naturally be regarded as an analogue of the Ahlfors-Heins theorem for quasianalytic functions. It shows that the asymptotics of functions of the class $C\{M_n\}$ is identical with the asymptotics along the real axis of functions of the class A in the strip $D_a(F)$.

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* Theorem 3 sharpens a result obtained by Hirschman (([7]), Theorem 3a).

Note: Figure translations are in progress. See original paper for figures.