FUNCTIONS OF ORDERED SELECTION OF CONTINUOUS VARIABLES

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Abstract

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FUNCTIONS OF ORDERED SELECTION OF CONTINUOUS VARIABLES

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The apparatus of continuous logic has in recent years been applied ever more widely in the construction of analog converters, restoring devices, in the analysis and synthesis of nonlinear electrical networks, and also in solving other problems in the design of automatic-control systems and information-processing devices ([1–6]). In this connection the simplest functions, max, min, and inv, are most often used, and only recently has the median function been defined, which is an analogue of the majority voting function ([3]). In the present paper definitions are given and some of the most important properties are considered for two other functions of continuous logic that generalize to the continuous case the voting function "m out of n" and the threshold function of binary variables.

Let there be n continuous variables $x_1, x_2, \dots, x_n$, where $A \le x_i \le B$, $i = 1, \dots, n$. Any set of fixed values $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ of these variables can be ordered in nondecreasing or nonincreasing order. In the first case we shall agree to denote the ordered sequence of values of the variables by $\vec{x}$, and in the second by $\overleftarrow{x}$.

Definition 1. A **function of ordered selection** $M_n^m\{\vec{x}\}$, or $M_n^m\{\overleftarrow{x}\}$, where m is a given integer, $1 \le m \le n$, is a function, defined for all $A \le x_i \le B$, $i = 1, \dots, n$, which assigns to each $\tilde{x} = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\}$ the value $\tilde{x}_{i_j}(\tilde{x}_{j_s})$, occupying the m-th place in the ordered sequence $\{\vec{x}\}$ (or $\{\overleftarrow{x}\}$).

It is obvious that for a given $\tilde{x} = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\}$

$$M_n^m\{\vec{x}\} = M_n^{n-m+1}\{\overleftarrow{x}\}, \qquad M_n^m\{\overleftarrow{x}\} = M_n^{n-m+1}\{\vec{x}\}. \tag{1}$$

We note that in the case $m = (n+1)/2$ we obtain the median function ([6]):

$$M_n^{(n+1)/2}\{\vec{x}\} = M_n^{(n+1)/2}\{\overleftarrow{x}\} = \mathrm{med}\{x\}.$$

If the x_i take only discrete values 0 or 1, then the ordered-selection function $M_n^m\{\overleftarrow{x}\}$ is the voting function "m out of n."

The ordered-selection function can be represented as a superposition of the elementary functions max and min of continuous logic. Let us compose the min functions of all possible combinations of n variables taken m at a time:

$$\begin{aligned} u_1 &= \min(x_1, x_2, \dots, x_{m-1}, x_m), \\ u_2 &= \min(x_1, x_2, \dots, x_{m-1}, x_{m+1}), \\ &\dots\dots\dots\dots \\ u_r &= \min(x_{n-m+1}, x_{n-m+2}, \dots, x_{n-1}, x_n), \end{aligned} \tag{2}$$

where $r = C_n^m$. From (2) and the definition of the ordered-selection function it follows that

$$M_n^m\{\bar{x}\} \geq u_1, \quad M_n^m\{\bar{x}\} \geq u_2, \ \dots, \quad M_n^m\{\bar{x}\} \geq u_r, \tag{3}$$

for, if at least one $u_i > M_n^m\{\vec{x}\}$ were found, this would mean that there are m variables taking values strictly greater than $M_n^m\{\vec{x}\}$, which is impossible. From (3) it follows that

$$M_n^m\{\vec{x}\} \geq \max_{1 \leq i \leq r} u_i. \tag{4}$$

On the other hand, at each point $\tilde{x} = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\}$, $A \leq x_i \leq B$, $i = 1, \dots, n$, there are at least m variables whose values are greater than or equal to $M_n^m\{\vec{x}\}$; therefore there will be such a u_i that $M_n^m\{\vec{x}\} \leq u_i$, and hence

$$M_n^m\{\vec{x}\} \leq \max_{1 \leq i \leq r} u_i. \tag{5}$$

It follows from (4) and (5) that

$$M_n^m\{\vec{x}\} = \max_{1 \leq i \leq r} u_i. \tag{6}$$

Forming the max over all possible combinations of n variables taken m at a time:

$$\begin{aligned} v_1 &= \max(x_1, x_2, \dots, x_{m-1}, x_m), \\ v_2 &= \max(x_1, x_2, \dots, x_{m-1}, x_{m+1}), \\ &\dots\dots\dots\dots \\ v_r &= \max(x_{n-m+1}, x_{n-m-2}, \dots, x_{n-1}, x_n), \end{aligned} \tag{7}$$

one can show analogously that

$$M_n^m\{\vec{x}\} = \min_{1\leq i\leq r} v_i. \tag{8}$$

The number of logical elements required to realize the functions $M_n^m\{\vec{x}\}$ and $M_n^m\{\vec{x}\}$ is equal to $C_n^m + 1$. But in the case $m < (n+1)/2$ it is advantageous (in order to reduce the required number of elements) to use (1). Then it is easy to see that, for realizing $M_n^m\{\vec{x}\}$ and $M_n^m\{\vec{x}\}$, $C_n^{n-m+1}+1$ logical elements are sufficient.

The ordered-choice function $M_n^m\{\vec{x}\}$ can be generalized by assigning to the variables $x_1, x_2, \dots, x_n$ weight coefficients $a_1, a_2, \dots, a_n$, where a_i $(i = 1, \dots, n)$ are natural numbers.

Definition 2. An **ordered-choice function with preference** $F_n^m\{a,\vec{x}\}$ (or $F_n^m\{a,\vec{x}\}$), where m is a given integer, $1 \leq m \leq N = \sum_{i=1}^n a_i$, is a function of the variables $x_1, x_2, \dots, x_n$, with weights $a_1, a_2, \dots, a_n$ assigned to them, defined on the domain $A \leq x_i \leq B$, $i = 1, \dots, n$, and assigning to each set of values of the variables $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ the value occupying the k-th place in the ordered sequence

$$x = (\tilde{x}_{i_1} \geq \tilde{x}_{i_2}, \dots, \tilde{x}_{i_n})$$

(or

$$x = \tilde{x}_{j_1} \leq \tilde{x}_{j_2}, \dots, \tilde{x}_{j_n})$$

of these values, where k is determined from the condition

$$\sum_{s=1}^{k} a_{i_s} \geq m > \sum_{s=0}^{k-1} a_{i_s} \qquad (a_{i_0} = 0). \tag{9}$$

The name we have adopted for this function is explained by the fact that the values of the variable having the greater weight are given "preference" in the choice, in the sense that, for identical distribution functions, its values are selected more often.

The ordered-choice function with preference defined in the indicated way is a generalization to the case of continuous variables of the threshold function of binary variables. Indeed, assuming,

that all $\dot{x}_i$ take only the discrete values 0 or 1, from (9) we have

$$F_n^m\{a,\vec{x}\} = \begin{cases} 0 & \text{if } \sum_{i=1}^n a_i x_i < m, \\ 1 & \text{if } \sum_{i=1}^n a_i x_i \geqslant m, \end{cases} \tag{10}$$

which is also the definition of the threshold function (7).

If the variables $x_1, x_2, \dots, x_n$ with integer weight coefficients $a_1, a_2, \dots, a_n$ are considered as an expanded vector of variables $[a \otimes x] = \{x_1, \dots, x_n\}$, where each variable x_i is repeated as many times as its weight, then between the ordinary ordered-choice function and the function $F_n^m\{a, x\}$ the following relation holds:

$$F_n^m\{a, \vec{x}\} = M_N^m\{[(\vec{a} \otimes x)]\}. \tag{11}$$

Using relations (6) or (8) and (11), one can find a representation of the function $F_n^m\{a, \vec{x}\}$ in terms of max and min.

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Note: Figure translations are in progress. See original paper for figures.