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Abstract

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MATHEMATICS

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COERCIVE BOUNDARY-VALUE PROBLEMS FOR DIFFERENCE OPERATORS

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1. The concept of ellipticity of difference operators (see ([1,2])) plays, in the general theory of difference schemes, a role analogous to that of the well-known condition of I. G. Petrovskii for differential operators. It makes it possible to construct difference approximations that preserve on the grid the basic properties inherent in elliptic differential equations. However, the elliptic theory of difference operators cannot be regarded as sufficiently complete without considering difference boundary-value problems and establishing a discrete analogue of the well-known Shapiro–Lopatinskii condition of coercivity of boundary conditions. The necessity of such an algebraic condition is dictated by the needs of applied analysis, since it is important to have a simple criterion for estimating the stability of various difference approximations of elliptic boundary-value problems near the boundary of the domain. The present note is devoted precisely to this question.

2. **The space of grid functions $H_s^*(R_{n,h}^+)$.** By $R_{n,h}$ we shall denote the uniform grid with mesh size h in the Euclidean space R_n; correspondingly, by $R_{n,h}^+$ we shall denote the grid half-space:

$$R_{n,h}^+ = \{x \mid x \in R_{n,h}, \quad x_n \geq 0\}. \tag{1}$$

We shall consider grid functions on $R_{n,h}$ (respectively on $R_{n,h}^+$) with values in C^1, depending continuously on the grid parameter h, $0 \leq h \leq h_0$; everywhere in what follows h_0 is fixed and sufficiently small. For the definition and properties of the spaces of grid functions $H_s^* = H_s^*(R_{n,h})$ for any real s, see ([2]). We now define the spaces of grid functions $H_s^*(R_{n,h}^+)$. Denote by $\bar{H}_s^*$ the closed subspace of functions from H_s^* that are equal to zero for $x \in R_{n,h}^+$. Put

$$H_s^*(R_{n,h}^+) = H_s^*/\bar{H}_s^*, \tag{2}$$

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and $H_s^*(R_{n,h}^+)$ is endowed with the topology corresponding to the quotient norm, which we shall denote by $+\|\cdot\|_s^*$.

3. **Factorization of a quasihomogeneous function.** In formulating boundary-value problems for difference operators in convolutions (r.o.s.; see ([2])), the problem naturally arises of factorizing, with respect to one of the variables, functions defined on the torus. Let

$$T^n = \{\omega \mid \omega = (\omega_1, \dots, \omega_n),\ |1 + i\omega_k| = 1,\ 1 \le k \le n\}$$

be the n-dimensional torus. We shall consider scalar functions $\tilde{a}(x;\omega)$, defined on $\Omega \times T^n$, where Ω is a domain in $R_{n,h}$. We shall assume that:
1) $\tilde{a}$ is infinitely differentiable with respect to ω for $\omega \ne 0$;
2) $\tilde{a}$ is continuous in the aggregate of its arguments for $\omega \ne 0$;
3) $\tilde{a}$ satisfies the inequalities:

$$|\tilde{a}(x;\omega)| \le c_0 |\omega|^\alpha$$

(quasihomogeneity) and

$$|\tilde{a}(x;\omega)| \ge c_1 |\omega|^\alpha$$

(ellipticity) (see ([2]));
4) $\tilde{a}(x;\omega) = \tilde{a}(\infty;\omega) + \tilde{a}'(x;\omega)$, where $\tilde{a}'(x;\omega)$, as a function of the first argument, has compact support and belongs to the space-

space S^* (the lattice analogue of the Schwartz space S, see (2)). The class of such functions $\tilde{a}_\alpha$ will be denoted by $\mathcal{E}_\alpha$.

Put

$$D^+ = \{\omega_n \mid |1 + i\omega_n| > 1\}, \qquad D^- = \{\omega_n \mid |1 + i\omega_n| < 1\},$$

and by Γ we shall denote the boundary of the domains:

$$\Gamma = \{\omega_n \mid |1 + i\omega_n| = 1\}.$$

By a quasihomogeneous factorization with respect to ω_n of a function $\tilde{a}_\alpha \in \mathcal{E}_\alpha$ we shall mean a representation of $\tilde{a}_\alpha$ in the form

$$\tilde{a}_\alpha = \tilde{a}_\chi^+ \cdot \tilde{a}_{\alpha-\chi}^-, \tag{3}$$

where $\tilde{a}_\chi^+ \in \mathcal{E}_\chi$, $\tilde{a}_{\alpha-\chi}^- \in \mathcal{E}_{\alpha-\chi}$, $\tilde{a}_\chi^+$ (respectively, $\tilde{a}_{\alpha-\chi}^-$) is analytic in ω_n for $\omega_n \in D^+$ for all $\omega' \in T^{n-1}$ (respectively, for $\omega_n \in D^-$ for all $\omega' \in T^{n-1}$); moreover, $\tilde{a}_\chi^+$, for arbitrary finite $\omega_n \in D^+$, satisfies the inequalities

$$c_1 |\omega|^\chi \le |\tilde{a}_\chi^+| \le c_2 |\omega|^\chi, \qquad \forall\, \omega' \in T^{n-1}; \tag{4}$$

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respectively, $\tilde{a}^-_{\alpha-\chi}$ satisfies the inequalities

$$c_3|\omega|^{\alpha-\chi} \le |\tilde{a}^-_{\alpha-\chi}| \le c_4|\omega|^{\alpha-\chi}, \qquad \forall\, \omega_n \in D^-, \quad \forall\, \omega' \in T^{n-1}. \tag{5}$$

Theorem 1. *If $\tilde{a}_\alpha \in \mathcal{E}_\alpha$, then $\tilde{a}_\alpha$ admits a quasihomogeneous factorization* (3) *with respect to ω_n. The quasihomogeneous factorization* (3) *for $\tilde{a}_\alpha$ is unique up to a factor depending only on ω' and different from zero for all $\omega' \in T^{n-1}$.*

Let us note that for $\omega' \in T^{n-1} \setminus \{0\}$, for $\tilde{a}_\alpha \in \mathcal{E}_\alpha$ there is defined

$$\operatorname{ind} \tilde{a}_\alpha = (2\pi)^{-1}[\arg \tilde{a}_\alpha]_\Gamma, \tag{6}$$

which, by virtue of the connectedness of the set $T^{n-1} \setminus \{0\}$, does not depend on ω'. It is obvious that for all $\omega' \in T^{n-1} \setminus \{0\}$

$$\operatorname{ind} \tilde{a}_\alpha = \operatorname{ind} \tilde{a}^+_\chi = m, \tag{7}$$

but, generally speaking, the integer m does not coincide with the order of quasihomogeneity χ of the function $\tilde{a}^+_\chi$. We shall assume that m and χ do not depend on $x \in \Omega$.

4. Smooth operators. In what follows, an essential role will be played by the following

Condition (*). A quasihomogeneous function $\tilde{a}_\alpha$ of order α satisfies condition $(*)$ if, uniformly with respect to $x \in \Omega$ and for all integers k, $k \ge 0$, the relations

$$\int_\Gamma \omega_n^k \tilde{a}_\alpha \, d\omega_n = O(|\omega'|^{\alpha+k+1}) \quad \text{as } \omega' \to 0, \quad \omega' \in T^{n-1} \setminus \{0\}. \tag{8}$$

hold.

Of course, (8) can be rewritten in terms of equality to zero of the integrals over Γ of the product of ω_n^k by the corresponding derivatives with respect to ω' of $\tilde{a}_\alpha$ at the point $\omega' = 0$.

Theorem 2. *Let $\tilde{a}_\alpha \in \mathcal{E}_\alpha$, and let* (3) *be the corresponding quasihomogeneous factorization of the function $\tilde{a}_\alpha$. Suppose that $\tilde{a}_\alpha$ and $\tilde{a}_\alpha^{-1}$ satisfy condition $(*)$. Then*

$$m = \operatorname{ind} \tilde{a}^+_\chi = \chi, \qquad \forall\, \omega' \in T^{n-1} \setminus \{0\}. \tag{9}$$

To the quasihomogeneous function $\tilde{a}_\alpha(x;\omega)$ one can assign a generalized homogeneous function of $(h, \zeta, \bar{\zeta})$ of order α:

$$\tilde{a}_\alpha(x; h, \zeta, \bar{\zeta}) = h^{-\alpha}\tilde{a}_\alpha(x; h\xi), \qquad \zeta = (\zeta_1, \dots, \zeta_n), \quad \zeta_k = (e^{ih\xi_k} - 1)/ih. \tag{10}$$

The function $\tilde{a}_\alpha(x; h, \zeta, \bar{\zeta})$ defines on lattice functions in $R_{n,h}$ a homogeneous difference operator in convolutions (h.d.o.c.) A_α, acting continuously from H^*_s to $H^*_{s-\alpha}$ for any $s \in R^1$ (see (2)).

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Let a lattice function $u_+ \in H_s^*(R_{n,h}^+)$; denote by $l_0 u_+$ the lattice function in $R_{n,h}$, equal to u_+ for $x \in R_{n,h}^+$ and to zero for $x \notin R_{n,h}^+$, and, so-

respectively, by $+H_s^*$ the space of all such extensions. Conversely, if a mesh function u is given on $R_{n,h}$, then we denote by R_+ the operator of restriction of u to $R_{n,h}^+$: $R_+ u = u_+$. Naturally, the question arises as to when $R_+ A_\alpha l_0 u_+ \in H_{s-a}^*(R_{n,h}^+)$? As simple examples show (for example, the shift operator with respect to x_n), such an inclusion does not always hold. However, the following is true.

Theorem 3. *Let the quasi-homogeneous function $\tilde{a}_\alpha$ satisfy condition $(*)$. Then the homogeneous d.o. A_α corresponding to it acts continuously from $+H_s^*$ into $H_{s-\alpha}(R_{n,h})$.*

5. Formulation of the boundary-value problem in the case of an integer $\varkappa \geq 0$. Let A be a d.o. of order a with complete symbol

$$\sigma(A) \sim \sum_{k=0}^{\infty} \tilde{a}_k(x; h, \xi, \bar{\xi}), \tag{11}$$

satisfying the ellipticity condition. Suppose that the number $\varkappa$ in the factorization (3) of the function

$$\tilde{a}(x; \omega) = \tilde{a}_0(x; 1, \omega, \bar{\omega}) \tag{12}$$

is an integer and $\varkappa \geq 0$. Let, moreover, $\tilde{a}_k(x; 1, \omega, \bar{\omega})$, $k \geq 0$, and $\tilde{a}^{-\tau}(x; \omega)$ satisfy condition $(*)$. We prescribe $\varkappa$ d.o.' s B_j of orders α_j, $1 \leq j \leq \varkappa$, with complete symbols

$$\sigma(B_j) \sim \sum_{k=0}^{\infty} \tilde{b}_{kj}(x; h, \xi, \bar{\xi}). \tag{13}$$

Suppose that the functions $\tilde{b}_{kj}(x; 1, \omega, \bar{\omega})$ satisfy condition $(*)$, and set

$$\tilde{c}_{jl}(x; \omega') = (2\pi i)^{-1} \int_\Gamma \tilde{b}_{0j} \omega_n^{l-1} (\tilde{a}_x^+)^{-1} \, d\omega_n, \qquad 1 \leq j, l \leq \varkappa, \quad \omega' \in T^{n-1} \setminus \{0\}, \tag{14}$$

where $\tilde{a}_x^+$ is the factor analytic for $\omega_n \in D^+$ in the factorization (3) of the function $\tilde{a}(x; \omega)$ from (12). It is not difficult to verify that $\tilde{c}_{jl}(x; \omega')$ is a quasi-homogeneous function of ω' of order $\alpha_j + l - \varkappa$. On the matrix $\|\tilde{c}_{jl}\|$ we impose the following basic condition (an analogue of the Shapiro–Lopatinskii condition for differential elliptic boundary-value problems):

Condition $(**)$**.** The matrix $\|\tilde{c}_{jl}\|$ satisfies the inequality

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$$\left|\det \|\tilde{c}_{jl}\|\right| \geq c|\omega'|^{p_0}, \qquad \omega' \in T^{n-1} \setminus \{0\}, \quad x \in R^+_{n,h}, \tag{15}$$

where

$$p_0 = \sum_{j=1}^{\varkappa} \alpha_j - \varkappa(\varkappa - 1)/2,$$

and c is a constant independent of x and ω'.

Boundary operators satisfying conditions $(*)$ and $(**)$ will be called coercive. To the operators A and $\{B_j\}$, $1 \leq j \leq \varkappa$, we associate the boundary-value problem in $R^+_{n,h}$:

$$R_+ A u_+ = f, \qquad f \in H^*_{s-a}(R^+_{n,h}), \tag{16}$$

$$R_+ B_{ju} + \big|_{x_n=0} = \varphi_j, \qquad \varphi_j \in H^*_{s-\alpha_j-1/2}(R_{n-1,h}), \qquad 1 \leq j \leq \varkappa. \tag{17}$$

Theorem 4. *Let all the conditions listed above on the operators A and B_j, $1 \leq j \leq \varkappa$, be satisfied. Then for a solution u_+ of the boundary-value problem (16), (17) the a priori estimate*

$$^+\|u_+\|^*_s \leq c \left(^+\|f\|^*_{s-a} + \sum_{j=1}^{\varkappa} {}'\|\varphi_j\|^*_{s-\alpha_j-1/2} + {}^+\|u_+\|^*_{s-1} \right), \tag{18}$$

holds, where $'\|g\|^*_s$ is the norm of $g(x')$ in $H^*_s(R_{n-1,h})$, and c is independent of h.

A converse assertion is also true in a certain sense.

Theorem 5. If, for the boundary-value problem (16), (17), for some s the estimate (18) holds for arbitrary $f \in \dot{H}^*_{s-\alpha}(R^+_{n,h})$ and $\varphi_j \in \dot{H}^*_{s-\alpha_j-1/2}(R_{n-1,h})$, and the operators A, B_j, $1 \leq j \leq \chi$, satisfy condition $(*)$, then the difference operator A is elliptic, and the boundary difference operators B_j, $1 \leq j \leq \chi$, satisfy the coercivity condition (15).

6. Formulation of the boundary-value problem in the case of an integer $\chi < 0$. In this case every solution of equation (16) with smooth right-hand sides will contain a nonsmooth part of the form

$$\sum_{j=1}^{|\chi|} \rho_j(x') D_n^{j-1} \delta(x_n), \tag{19}$$

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where $\delta(x_n)$ is the lattice δ-function, $\delta(x_n) = h^{-1}$ for $x_n = 0$ and 0 for $x_n \neq 0$. It is natural to divide the problem of finding a solution of equation (16) into the problem of finding the part u_+, smooth up to the boundary, and the nonsmooth summands $\rho_j(x')D_n^{j-1}\delta(x_n)$, concentrated in the boundary strip $0 \leq x_n \leq (j-1)h$, $1 \leq j \leq |\chi|$. Therefore, for $\chi < 0$, the following problem is posed: to find a solution u_+, smooth up to the boundary, and potentials ρ_j, $1 \leq j \leq |\chi|$, satisfying the equation

$$R_+\left(Au_+ + \sum_{j=1}^{|\chi|} G_j\rho_j\right) = f, \tag{20}$$

where G_j are difference operators of orders α_j satisfying condition $(*)$, as well as a coercivity condition analogous to $(**)$. In this case there is also a corresponding a priori estimate for $u_+(x)$ and the densities $\rho_j(x')$.

In conclusion, we note that, for reasonable difference approximations of elliptic boundary-value problems for differential operators, condition $(*)$ is always satisfied (possibly after applying some power of the shift operator in the normal-to-the-boundary variable, which means merely another writing of the difference equations in an equivalent form). However, condition (15) does not always hold; in that case, as was already said above, an a priori estimate of type (18) is invalid, and near the boundary of the domain, as a rule, instability arises in numerical experiments. Therefore the algebraic condition (15) is in practice an effective way of checking the soundness of a difference approximation near the boundary of an elliptic boundary-value problem, just as the ellipticity condition for a difference operator is a criterion of a good approximation of an elliptic differential operator inside the domain.

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