A CRITERION FOR REPRESENTABILITY OF A DIRECT WREATH PRODUCT OF GROUPS BY MATRICES

1970

SovietRxiv

Abstract

Full Text

UDC 519.45+512.86

MATHEMATICS

Yu. E. VAPNE

A CRITERION FOR REPRESENTABILITY OF A DIRECT WREATH PRODUCT OF GROUPS BY MATRICES

(Presented by Academician P. S. Novikov on 6 IV 1970)

The purpose of the present note is to prove the following two propositions.

Theorem 1. Let A, B be nontrivial groups isomorphically representable by matrices over a field of characteristic 0. The direct wreath product $W = A \wr B$ is isomorphically representable by matrices over a field of characteristic 0 if and only if one of the following conditions is satisfied:

1) B is a finite group,

2) B is a finite extension of a torsion-free abelian group, and A is a torsion-free abelian group.

Theorem 2. Let A, B be nontrivial groups isomorphically representable by matrices over a field of characteristic $p > 0$. The direct wreath product $W = A \wr B$ is isomorphically representable by matrices over a field of characteristic p if and only if one of the following conditions is satisfied:

1) B is a finite group,

2) B is a finite extension of a torsion-free abelian group, and A is an abelian p-group of finite period.

These propositions give the criterion mentioned in the title and thereby solve, for wreath products, question 3.29 from ([1]). In paper ([2]), Theorems 1, 2 were proved by the author for the case when the active group B is almost solvable. In the present note the general case is reduced to this particular case. Further, from Chevalley' s results on semisimple algebraic groups of matrices $(({}^3), pp.23–02)$ it follows that for matrix groups over a field, almost solvability is equivalent to the satisfaction of a nontrivial identity ([4]). Therefore Theorems 1, 2 follow from ([2]) and Lemmas 3, 4 proved below.

Let R be a ring without zero divisors, $R[G]$ the group ring of the group G over R. If m is a natural number and γ is an element of $R[G]$ equal to

$$\sum_{i=1}^{n} r_i g_i,$$

where r_i are nonzero elements of R, $g_i \in G$, then let $Q(m, \gamma)$ denote the set of all possible products of at most m elements of the set

$$\{g_2 g_1^{-1},\ g_3 g_1^{-1}, \dots,\ g_{ng} 1^{-1}\}.$$

We also denote

$$Q(\gamma) = \bigcup_{m=1}^{\infty} Q(m, \gamma).$$

The sets $Q(m, \gamma)$, $Q(\gamma)$ in fact depend not on the element γ, but on its expression. From the context it will always be clear which expression is meant.

Lemma 1. Let $\alpha, \beta \in R[G]$, $\alpha \neq 0$, and $\alpha\beta = 0$, and let m be the number of summands in an irreducible expression for α. Then the set $Q(m, \beta)$ contains the identity (independently of which expression for β is fixed).

Proof. Let

$$\alpha = \sum_i k_i a_i, \qquad \beta = \sum_j l_j b_j,$$

where k_i, l_j are nonzero elements of R, $a_i, b_j \in G$, and the elements a_i are pairwise distinct. By hypothesis,

$$\alpha\beta = \sum_{i,j} k_i l_j a_i b_j = 0.$$

Therefore

$$a_i b_1 = a_{\varphi(i)} b_{\psi(i)}$$

for

all i, where φ, ψ are functions, and $\psi(i) \neq 1$ for every i. Hence

$$a_{\varphi(i)}^{-1} a_i = b_{\psi(i)} b_1^{-1}. \tag{1}$$

Since φ maps the set $\{1, 2, \dots, m\}$ into itself, there exist elements s, t of this set such that $\varphi^s(t) = t$. Multiplying relations (1) for $i = \varphi(t), \varphi^2(t), \dots, \varphi^s(t)$, we obtain the identity on the left-hand side, and on the right an element of $Q(m, \beta)$.

The lemma is proved.

Let F be the free group with free generators $x_1, x_2, \dots, \dots, x_r, y_1, y_2, \dots$. Let R be a ring with identity. Consider in $R[F]$ the element

$$\xi = \sum_{\pi} \operatorname{sgn} \pi \cdot x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(r)}, \tag{2}$$

where the summation is over all permutations π on r symbols, and $\operatorname{sgn} \pi$ is the sign of the permutation π.

Lemma 2. *The set $Q(\xi)$ does not contain the identity. In particular, no simple commutator of elements of the set $Q(\xi)$, successively conjugated by the elements $y_1, y_2, \dots$, is equal to the identity.*

Proof. We shall prove only the first assertion; the second is its immediate consequence.

Suppose, on the contrary, that some element

$$x_{\pi_1(1)} \cdots x_{\pi_1(r)} x_r^{-1} \cdots x_1^{-1} \cdots x_{\pi_n(1)} \cdots x_{\pi_n(r)} x_r^{-1} \cdots x_1^{-1}$$

from $Q(\xi)$, where $\pi_1, \dots, \pi_n$ are nonidentity permutations, is equal to the identity. This means that there exists a sequence of cancellations which transforms the above word into the empty word. At some step of this sequence the first occurrence of the element $x_{\pi_1(i)}$, $1 \le i \le r$, must cancel with some occurrence of the element $x_{\pi_1(i)}^{-1}$; then the word w_i standing between these occurrences must have become the empty word at preceding steps. It is easy to compute that $\log w_i = \pi_1(i) - i$, where $\log w_i$ is the sum of the exponents of all letters occurring in w_i. But, as we have noted, $w_i = 1$, hence $\log w_i = 0$. Therefore $\pi_1(i) = i$, i.e. the permutation π_1 is the identity. The contradiction obtained proves the lemma.

Lemma 3. *Let $W = (a) \wr G$, where (a) is an infinite cyclic group and G is a group. If the wreath product W is isomorphically representable by matrices over a field of characteristic 0, then a nontrivial identity holds on G.*

Proof. Suppose the wreath product W is isomorphically represented by matrices over an algebraically closed field k of characteristic 0.

First note that the matrix a may be assumed unipotent. If the group G is periodic, then by Schur' s theorem an identity holds on it (([5], p. 533)). Let G be nonperiodic and let g be an element of infinite order. The subgroup (a, g) of the group W is isomorphic to the wreath product of two infinite cyclic groups and hence is solvable. By Mal' cev' s theorem (([5], p. 535)) the group (a, g) is almost triangularizable; we shall assume it almost triangular. For some n the matrices a^n and g^n are triangular; therefore their commutator $c = [a^n, g^n]$ is unipotent. By Lemma 12 from ([2]) the subgroup (c, G) is isomorphic to W.

Thus, we shall assume that the matrix a is unipotent. Then the entire base F of the wreath product W is unipotent. There exists a nonzero vector v of the underlying space fixed by F. Let U be the subspace spanned by the G-orbit of the vector v. Obviously, U is invariant under G and fixed by F, since

$$vgf = vgfg^{-1}g = vf^{g^{-1}}g = vg, \quad \text{where } f \in F,\ g \in G.$$

This means that, with a suitable choice of basis of the underlying space, the elements of the group W have the form

$$w = \begin{pmatrix} \varphi_{11}(w) & \varphi_{12}(w) \\ 0 & \varphi_{22}(w) \end{pmatrix},$$

where $\varphi_{22}(F) = e$, with e the identity matrix. Let K be the kernel of the homomorphism φ_{11}. If K is trivial, induction on the degree of the matrices completes the proof. Let K be nontrivial. Since a nontrivial normal subgroup of a wreath product has nontrivial intersection with the base (see, for example, Lemmas 8.1 and 8.2 of ([6])), it follows that $K \cap F \neq 1$. Fix some nonidentity element f of $K \cap F$. Obviously, $\varphi_{11}(f) = e,\ \varphi_{22}(f) = e$.

Let V be the linear space of all matrices over the field k having the same dimensions as the matrix $\varphi_{12}(f)$. It is easy to see that the formula

$$u \sum n_i g_i = \sum n_i \varphi_{11}(g_i)^{-1} u \varphi_{22}(g_i), \qquad n_i \in Z, \quad g_i \in G, \quad u \in V,$$

defines an action of the ring $Z[G]$ on the space V. This action determines a ring homomorphism $Z[G] \to \operatorname{End} V$, where $\operatorname{End} V$ is the algebra of endomorphisms of the space V over the field k. Let I be the kernel of this homomorphism.

We shall naturally regard the base F of the wreath product W as an exact cyclic right $Z[G]$-module (see, for example, ([2])). Let $f = a^\alpha,\ \alpha \in Z[G]$. By assumption $\alpha \neq 0$. If $\beta \in I$, then

$$a^{\alpha\beta} = f^\beta = \begin{pmatrix} e & \varphi_{12}(f)\beta \\ 0 & e \end{pmatrix} = 1,$$

whence $\alpha\beta = 0$. Thus, all elements of I annihilate α.

The factor ring $Z[G]/I$ is isomorphically embedded in the algebra $\operatorname{End} V$. Let $(r-1)$ be the dimension of this algebra over k. According to ([7], p. 329), the standard identity $\xi = 0$ holds on it, where ξ is the element from (2). Then $\beta = \xi(g_1, \dots, g_r) \in I$ for any $g_1, \dots, g_r \in G$. As we noted above, $\alpha\beta = 0$. Let m be the number of terms in the reduced expression of the element α. By Lemma

1 the set $Q(m, \beta)$ contains the identity. Let $q_1, \dots, q_s$ be all the distinct elements of $Q(m, \beta)$. Obviously, s depends only on f and r, and $q_i = w_i(g_1, \dots, g_r)$, where $w_i \in Q(m, \xi)$. If $h_1, \dots, h_s$ are arbitrary elements of G, then the simple commutator

$$[q_1^{h_1}, q_2^{h_2}, \dots, q_s^{h_s}] = 1.$$

We have obtained an identity on the group G. By Lemma 2 it is nontrivial.

Lemma 4. Let $W = (a) \wr G$, where (a) is a cyclic group of prime order p, and let G be a group. If the wreath product W is isomorphically representable by matrices over a field of characteristic p, then a nontrivial identity holds on G.

Proof differs hardly at all from the proof of Lemma 3. It is only necessary to note that now the matrix a is unipotent by assumption (its order is equal to p), and instead of $Z[G]$ one must everywhere consider the group ring over the prime field of characteristic p.

In view of Lemma 3, Theorem 1 is now a consequence of the analogous Theorem 1 and Lemma 14 from ([2]). In exactly the same way, in view of Lemma 4, Theorem 2 is a consequence of Theorem 2 and Lemma 15 from ([2]). Let us note in passing that, in the proof of Lemma 15, the contradiction with the fact that the element a has infinite order is obtained more briefly and neatly as follows. Since the matrices a^b, $b \in B$, belong to the diagonal group F_0 and have the same eigenvalues as the matrix a, the number of these matrices is finite. On the other hand, they are pairwise distinct in the wreath product W_0—a contradiction.

In conclusion, the author expresses gratitude to Yu. I. Merzlyakov for his constant attention and useful discussions.

Novosibirsk State
University

Received
26 III 1970

REFERENCES

1. *The Kourovka Notebook. Unsolved Problems in Group Theory*, 3rd ed., Novosibirsk, 1969.
2. Yu. E. Vanya, *Mathematical Notes*, **7**, No. 2, 181 (1970).
3. C. Chevalley, *Classification des groupes de Lie algébriques*, Paris, 1958.
4. V. P. Platonov, *Dokl. AN BSSR*, **11**, No. 7, 581 (1967).

5. B. I. Plotkin, *Groups of Automorphisms of Algebraic Systems*, "Nauka," 1966.

6. R. M. Neumann, *Math. Zs.*, **84**, 343 (1964).

7. N. Jacobson, *Structure of Rings*, IL, 1961.

Note: Figure translations are in progress. See original paper for figures.