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MATHEMATICS

1970

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UDC 517.94

MATHEMATICS

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ON THE DIFFERENTIAL IRREDUCIBILITY OF A LINEAR INHOMOGENEOUS EQUATION

(Presented by Academician P. S. Aleksandrov on 20 III 1970)

Let k be a certain subfield of the field of analytic functions of the complex variable z, closed with respect to the operation of differentiation. Consider the equation

$$y^{(n)} + p_1 y^{(n-1)} + \ldots + p_{ny} = p_0, \qquad (1)$$

on whose left-hand side there stands a linear homogeneous differential polynomial $L(y)$ of order $n \geqslant 1$ with coefficients $p_1, \ldots, p_n \in k$, while the right-hand side p_0 also belongs to the field k. Equation (1) is called **differentially reducible** if it has a solution y_0, transcendental over k, satisfying an equation

$$R(y, y', \ldots, y^{(m)}) = 0, \qquad (2)$$

on whose left-hand side there stands a differential polynomial $R(y)$ (not necessarily linear) of order $m < n$ with coefficients from k. In the contrary case equation (1) is called **differentially irreducible**. It is clear that an equation of order $n = 1$ is always differentially irreducible. Therefore, in what follows we assume that $n \geqslant 2$. In applications, as the field k one usually considers the field of rational functions, which is evidently closed with respect to the operation of differentiation.

The notion of differential irreducibility of a linear equation plays an important role in the theory of transcendental numbers. In 1929 C. Siegel ([1]) created an analytic method that made it possible to reduce the question of the transcendence and algebraic independence of the values of one class of entire functions, called E-functions, to the investigation of the differential-algebraic properties of the functions themselves. The work begun by C. Siegel was completed by A. B. Shidlovskii. Shidlovskii' s theorem ([2]), as applied to an E-function $f(z)$ satisfying equation (1), asserts that the numbers $f(\alpha), f'(\alpha), \ldots, f^{(n-1)}(\alpha)$, for any algebraic $\alpha \neq 0$ distinct from the zeros of $p_0, p_1, \dots, p_n$, are algebraically independent if and only if the functions $f(z), f'(z), \dots, f^{(n-1)}(z)$ are algebraically independent over the field of rational functions. One of the ways of verifying this property of the function $f(z)$ consists in proving the differential irreducibility of equation (1).

As an application of his method, C. Siegel considered the functions

$$K_\lambda(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\lambda+1)\dots(\lambda+n)} \left(\frac{z}{2}\right)^{2n}, \qquad \lambda \neq -1, -2, \dots,$$

which are solutions of the differential equation

$$y'' + \frac{2\lambda+1}{z} y' + y = 0, \tag{3}$$

where λ is a complex parameter. He proved that if λ is not equal to half an odd number, then equation (3) is differentially irreducible. Using this assertion and the method indicated above, C. Siegel obtained the following theorem:

Let λ be a rational number not equal to half an odd number. Then the numbers $K_\lambda(\alpha)$ and $K'_\lambda(\alpha)$ are transcendental and algebraically independent for any algebraic $\alpha \neq 0$.

Subsequently, in works by A. B. Shidlovskii and other authors ([3–5]), a broader class of functions satisfying linear differential equations of the 2nd and 3rd orders was considered. Having established the differential irreducibility of these equations and using Shidlovskii' s theorem, it was possible to obtain a number of analogous theorems on the transcendence and algebraic independence of the values of the corresponding functions and their derivatives. Let us cite one of the results obtained by A. B. Shidlovskii in ([3]).

Consider the function

$$K_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda+1)\dots(\lambda+n)(\mu+1)\dots(\mu+n)} \left(\frac{z}{2}\right)^{2n},$$

which is a solution of the differential equation

$$y'' + \frac{2\lambda+2\mu+1}{z} y' + \left(1 + \frac{4\lambda\mu}{z^2}\right) y = \frac{4\lambda\mu}{z^2}. \tag{4}$$

Theorem 1. *If λ and μ are rational numbers distinct from negative integers, for which the difference $\lambda - \mu$ is not equal to half an odd number, and $\alpha \neq 0$ is any algebraic number, then the numbers $K_{\lambda,\mu}(\alpha)$ and $K'_{\lambda,\mu}(\alpha)$ are algebraically independent.*

The first and very important step in proving the differential irreducibility of a linear equation in the works cited is the passage from solutions of equation (1) to solutions of the homogeneous equation $L(y) = 0$. This passage subsequently makes it possible to solve the problem of the differential irreducibility of a linear homogeneous equation in the class of homogeneous algebraic differential equations $R(y) = 0$. The method used by R. Siegel carries out this passage only for equations of the 2nd order. In 1969, Yu. V. Nesterenko ([5]) showed that an analogous passage to homogeneous equations is possible also in the case of linear differential equations of arbitrary order. A substantial shortcoming of the theorems obtained is their irreversibility. This circumstance in most cases does not allow one to give a complete description of all parameter values for which the equation under study is differentially irreducible. In particular, Theorem 1 does not give a complete description of the values of λ, μ for which the numbers $K_{\lambda,\mu}(\alpha)$ and $K'_{\lambda,\mu}(\alpha)$ are algebraically independent.

In the present paper a theorem on a reversible passage to homogeneous equations will be formulated and proved. In applications to concrete equations of the 2nd order, this theorem makes it possible to give a complete description of the parameter values for which the equation under consideration is differentially irreducible.

Consider a linear homogeneous differential equation

$$L_1(y) = 0 \tag{5}$$

with coefficients from a field k of order $l \geq 2$. Equation (5) is called homogeneously reducible if it has a solution y_0, transcendental over k, satisfying the equation $Q(y) = 0$, on the left-hand side of which there stands a differential polynomial Q of order $m < l$ (not necessarily linear) with coefficients from k, homogeneous in the aggregate $y, y', \dots, y^{(m)}$.

In what follows we shall assume that in equation (1) the constant term $p_0 \neq 0$. Dividing both sides of the equation by this coefficient, differentiating with respect to z, and multiplying again by p_0, we obtain a linear homogeneous differential equation (5), the left-hand side of which has the form: $L_1(y) = p_0(p_0^{-1}L(y))' = y^{(n+1)} + \cdots$ and, consequently, is a non-identically-zero linear homogeneous differential polynomial of order $l = n + 1$. The equation obtained will be called the **enveloping** equation with respect to equation (1).

Remark 1. It is clear that all solutions of equation (1) satisfy its enveloping equation (5). Moreover, as is known, all solutions

equation (1) are described by the formula $c_1y_1 + \dots + c_{ny}n + y_{n+1}$, where $y_1, \dots, y_{n+1}$ are linearly independent particular solutions, and $c_1, \dots, c_n$ are arbitrary constants. Then it is not difficult to see that every solution of the encompassing equation (5) has the form $c_1y_1 + \dots + c_{ny}n + c_{n+1}y_{n+1}$, where $c_1, \dots, c_n, c_{n+1}$ are arbitrary constants. The indicated property of the encompassing equation may be taken as its definition.

Theorem 2. *The linear nonhomogeneous equation (1) is differentially reducible if and only if the encompassing equation (5) is homogeneously reducible.*

Proof. Necessity. Let equation (1) be differentially reducible, i.e., have a transcendental solution y_0 satisfying an equation (2) of order $m < n$. Without loss of generality, one may assume that the differential polynomial $R(y)$ standing on the left-hand side of equation (2) is irreducible (in the ordinary algebraic sense), as a polynomial in $y, y', \dots, y^{(m)}$ over the field k. Computing the total derivative of the polynomial $R(y)$ with respect to z, we obtain a differential polynomial $R'(y)$ of order $m+1$, whose root is y_0. It is not difficult to see that this polynomial is distinct from identically zero and in fact contains $y^{(m+1)}$. Since the degree of $R'(y)$ in the totality $y, y', \dots, y^{(m+1)}$ does not exceed the degree of the polynomial $R(y)$, which does not contain $y^{(m+1)}$, these two polynomials are relatively prime. Introduce the auxiliary function $u = y'y^{-1}$ and express, through $y, u, u', \dots, u^{(m)}$, all derivatives of y up to the $(m+1)$-st inclusive: $y' = uy$, $y'' = y(u^2 + u'), \dots$. It is easy to verify that the resulting transformation

$$y, y', \dots, y^{(m+1)} \to y, u, \dots, u^{(m)} \tag{6}$$

is polynomial. Passing by means of (6), in the polynomials $R(y)$ and $R'(y)$, from the functions $y, y', \dots, y^{(m+1)}$ to the functions $y, u, \dots, u^{(m)}$, we obtain two polynomials K and K_1 in $y, u, u', \dots, u^{(m)}$, which vanish under the substitution $y = y_0$, $u = u_0 = y_0' y_0^{-1}$. Note that $u, u', \dots, u^{(m)}$ can be represented in the form of rational functions of $y, y', \dots, y^{(m+1)}$, whose denominator depends only on y:

$$u = y'y^{-1}, \quad u' = [y''y - (y')^2]y^{-2}, \dots . \tag{7}$$

Consequently, transformation (6) is birational over the field k and bipolynomial over its transcendental extension $k[y]$. The first property of transformation (6) shows that $K \neq K_1 \neq 0$, since $R \neq R' \neq 0$. The second property shows that K and K_1 are relatively prime as polynomials in $u, u', \dots, u^{(m)}$ over the field $k[y]$. Consequently, the common factors of K and K_1, as polynomials in $y, u, u', \dots, u^{(m)}$, depend only on y. Since y_0 is a function transcendental over k, these factors cannot have y_0 as their root. Therefore, cancelling K and K_1 by the common factors, we obtain two polynomials over k, relatively prime in $y, u, \dots, u^{(m)}$, which vanish for $y = y_0$, $u = y_0' y_0^{-1}$. Eliminating y from these polynomials, we obtain a differential polynomial $\overline{Q}(u)$, not identically zero, of order $\leq m$, whose root is the function u_0. Note now that in the equalities (7) the right-hand sides are ratios of two homogeneous polynomials in $y, y', \dots, y^{(m+1)}$ of equal degrees, and the denominator is simply a power of y. Therefore, passing by means of (7) in the polynomial $\overline{Q}(u)$ to the unknowns $y, y', \dots, y^{(m+1)}$ and multiplying it by a suitable power of y, we obtain a differential polynomial $Q(y)$ of order $\leq m+1$, homogeneous in the totality $y, y', \dots, y^{(m+1)}$, whose root is the function y_0. Since $n > m$, the order of the encompassing equation (5),

$n+1 > m+1 \geq$ the order of the polynomial $Q(y)$. Since y_0 is a solution of equation (5), it is thereby proved that this equation is homogeneously reducible.

Sufficiency. Let the encompassing equation (5) be homogeneously reducible, i.e., have a transcendental solution y_0 satisfying a homogeneous differential equation $Q(y) = 0$ of order $m < n+1$.

By virtue of Remark 1 there exists a constant $c \neq 0$ such that the function cy_0 is a root of equation (1) ($c = c_{n+1}^{-1}$ for $c_{n+1} \neq 0$ and $c = 1$ for $c_{n+1} = 0$). Since the polynomial $Q(y)$ is homogeneous in the aggregate $y, y', \ldots, y^{(m)}$, cy_0 is also its root. Since equation (1) is nonhomogeneous, $p_0 \neq 0$, the polynomials $Q(y)$ and $L(y) - p_0$ are relatively prime over k. The order of each of them is $\leq n$. Eliminating $y^{(n)}$ from these two polynomials, we obtain a differential polynomial $R(y) \neq 0$ of order $< n$, whose root is cy_0. Consequently, equation (1) is differentially reducible. Theorem 2 is proved.

The formulation of the theorem just proved can be strengthened if one introduces the notion of reducibility classes. By the reducibility class D of equation (1) we shall mean the totality of all its transcendental solutions satisfying algebraic differential equations (2) of order $< n$. By the homogeneous reducibility class O of the enveloping equation (5) we shall mean the totality of all its transcendental solutions satisfying algebraic homogeneous differential equations of order $< n+1$.

Theorem 3. *The classes D and O coincide up to constant nonzero factors.*

Indeed, in proving the necessity in Theorem 2 it was shown that if $y_0 \in D$, then $y_0 \in O$. In proving sufficiency it was established that from $y_0 \in O$ it follows that $cy_0 \in D$, for $c \neq 0$.

Remark 2. Theorems 2 and 3 are proved under the assumption that equation (1) is nonhomogeneous, $p_0 \neq 0$. To study a homogeneous equation (1) by means of the theorems obtained, it is necessary to make the substitution $y \to y + p$, where $p \in k$ and is not a solution of equation (1). Then equation (1) passes into a nonhomogeneous one which, as is easy to verify, is differentially reducible if and only if the original homogeneous equation is differentially reducible.

Theorem 2 makes it possible to give a complete description of the values of the parameters λ, μ for which equation (4) is differentially irreducible. By Theorem 2, equation (4) is differentially reducible if and only if the enveloping equation is homogeneously reducible,

$$y''' + \frac{2\lambda + 2\mu + 3}{z} y'' + \left(1 + \frac{4\lambda\mu + 2\lambda + 2\mu + 1}{z^2}\right) y' + \frac{2}{z} y = 0.$$

Using the methods of the works ([4],[6]), one can show that the last equation is homogeneously reducible if and only if the pair λ, μ, up to permutation, has the form $n_1, n_2 + \frac{1}{2}$, where n_1, n_2 are integers. Hence it follows:

Theorem 4. *Equation (4) is differentially irreducible if and only if the pair λ, μ does not coincide with any of the pairs of numbers $n_1, n_2 + \frac{1}{2}$, where n_1, n_2 are integers.*

Using Shidlovskii' s theorem and Theorem 4, we obtain the following assertion:

Theorem 5. *For rational λ, μ satisfying the conditions of Theorem 4, the numbers $K_{\lambda,\mu}(\alpha)$ and $K_{\lambda\mu}(\alpha)$ are algebraically independent for every algebraic $\alpha \neq 0$.*

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Received
17 III 1970

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* This theorem was first obtained by I. I. Belogrivov ([7]) by an arithmetic method.