# BOUNDARY-VALUE PROBLEMS FOR SINGULAR ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS

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## Abstract

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# BOUNDARY-VALUE PROBLEMS FOR SINGULAR ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS

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At present, boundary-value problems for elliptic equations of higher order that degenerate on the boundary of the domain are being intensively studied ([3–5]).

In the present note one more class of elliptic operators of higher order is studied, which degenerate on the corresponding hyperplane. The case when such degeneration occurs in a tangent plane was studied by the author earlier ([2]).

1. Let $R_n^+$ denote the half-space $x_n = y > 0$ of the Euclidean $n$-dimensional space $R_n$ of points $x = (x', y)$, where $x' = (x_1, \dots, x_{n-1})$. We introduce for consideration a class of function spaces $H_\gamma^l$, a detailed description of which is given in ([1]).

Let $L$ be a linear differential operator of the form

$$L = \sum_{|l'|+2l_n \le r} a_{l',l_n}(x) D_{x'}^{l'} B_y^{l_n}, \tag{1}$$

where the coefficients $a_{l',l_n}$ are functions sufficiently smooth in $\overline{R_n^+}$, constant for sufficiently large values of $x$ in modulus. Here and below $B_y$ denotes the singular Bessel operator

$$\frac{\partial^2}{\partial y^2} + \frac{2\gamma}{y}\frac{\partial}{\partial y} \qquad (\gamma > 0).$$

On the operator $L$ at all points $x = x_0$ belonging to the domain $y \ge 0$ we impose the condition

$$\sum_{|l'|+2l_n = r} a_{l',l_n}(x)\xi^{l'}\xi_n^{2l_n} \ne 0 \qquad \text{for } |\xi| \ne 0. \tag{2}$$

Such operators were called $B$-elliptic operators by the author ([2]). We note that a condition of type (2) was first formulated by the author in ([2]). An operator $T$ is called a smoothing operator if it is a bounded operator from $H_\gamma^s$ to $H_\gamma^{s+1}$. Let first the operator $L$ have constant coefficients. Consider in all of $R_n^+$ the equation $Lu = L(D_{x'}, B_y)u(x) + f(x)$. Let $L_0$ be the principal part of the operator $L$.

**Theorem 1.** Let $u \in H_\gamma^s$, $s \geq 2m$, be a solution of the equation $L_0 u(x) = f(x)$. Then for $\gamma > 0$:

1) the a priori estimate is valid

$$\|u\|_{H_\gamma^s} \leq C\left(\|L_0 u\|_{H_\gamma^{s-2m}} + \|u\|_{H_\gamma^0}\right), \tag{3}$$

2) there exists a bounded operator $R_0$, acting from $H_\gamma^{s-2m}$ to $H_\gamma^s$, such that

$$L_0 R_0 = I + T. \tag{4}$$

With the aid of the preceding theorem the following is proved.

**Theorem 2.** Let $u \in H_\gamma^s$ and $s \geq 2m$. Let

$$L(x; D_{x'}, B_y) = \sum_{|l| \leq 2m} a_l(x) D_{x'}^{l'} B_y^{l_n}$$

be a $B$-elliptic operator with infinitely differentiable coefficients, bounded in all of $R_n^+$, and with higher coefficients satisfying the condition: there exists an $\varepsilon > 0$ such that for all $x \in \overline{R}_n^+$,

$$|a_l(x) - a_l(0)| < \varepsilon.$$

Let, moreover, $a_l(x)$ satisfy the condition

$$|D_{x',y}^{l'+k} a_l(x)| \leq C y^{2r-i-k}, \tag{5}$$

where $k + i \leq 2r,\ 1 \leq i \leq 2r-1,\ l' + 2r \leq s - 2m,\ |l| \leq 2m-1$. Then for $\gamma > 0$, for a solution of the equation $Lu = f$, the a priori estimate

$$\|u\|_{H_\gamma^s} \leq C\left(\|Lu\|_{H_\gamma^{s-2m}} + \|u\|_{H_\gamma^0}\right) \tag{6}$$

is valid, and there exists a bounded operator $R_B$, acting from $H_\gamma^{s-2m}$ into $H_\gamma^s$, such that

$$LR_B = I + T.$$

We now consider a linear differential operator of the form

$$L(x; D_{x'}, D_y^2) = \sum_{|l| \le 2m} a_l(x) D_{x'}^{l'} D_y^{2l_n}. \tag{7}$$

The conditions on the coefficients $a_l$ of the operator $L$ will be formulated below. We shall call the operator (7) $B$-elliptic if condition (2) is fulfilled for it in the domain $y \ge 0$. With the aid of Theorem 2 the following assertion is proved.

**Theorem 3.** Let $u \in H_\gamma^s$ and $s \ge 2m$. Let

$$L(x; D_{x'}, D_y^2) = \sum_{|l| \le 2m} a_l(x) D_{x'}^{l'} D_y^{2l_n} \tag{8}$$

be a $B$-elliptic operator with infinitely differentiable coefficients, bounded in $R_n^+$, and with higher coefficients satisfying the condition:

$$|a_l(x) - a_l(0)| < \varepsilon.$$

Let, moreover, $a_l(x)$ satisfy the conditions

$$\left| D_{x'}^{l'} \frac{\partial^k a_l(x)}{\partial y^k} \right| \le C y^{2r-i-k}, \tag{9}$$

$$k \le 2r - i, \qquad 1 \le i \le 2r - 1, \qquad l' + 2r \le s - 2m, \qquad |l| \le 2m - 1;$$

$$|a_l(x)| \le C_0 y^r, \qquad 1 \le r \le 2m - 1, \qquad |l| \le 2m - 1; \tag{10}$$

$$\left| D_{x'}^{l'} \frac{\partial^{i-k-r} a_l(x)}{(y\partial y)^{i-k-r}} \, y^{i-k-\tau} \right| \le C_1 y^{j+\tau}, \tag{11}$$

$$l' + i \le s - 2m, \qquad 1 \le j \le 2m - 1, \qquad |l| \le 2m, \qquad l_n \ne 0.$$

Then for $\gamma < 0$, for a solution of the equation $Lu = f$ in $R_n^+$, the estimate

$$\|u\|_{H_\gamma^s} \le C \left( \|Lu\|_{H_\gamma^{s-2m}} + \|u\|_{H_\gamma^0} \right) \tag{12}$$

is valid.

and there exists a bounded operator $R$ such that

$$LR = I + T, \tag{13}$$

where $R$ acts from $H_\gamma^{s-2m}$ into $H_\gamma^s$, and $T$ is a smoothing operator.

The results obtained in this part allow us to pass to a bounded domain.

2. Consider a bounded domain $\Omega$ in the space $R_n$. Let the boundary $\partial\Omega$ of this domain be a smooth manifold. Denote by $\{V_{\nu_1}\}$ a covering of the boundary $\partial\Omega$ by a finite system of domains $V_1, \dots, V_r$ in $R_n$ such that in each $V_k$ there is defined an infinitely smooth change of coordinates $x \to y = y^{(k)}$ with positive Jacobian, under which the boundary $\partial\Omega$ is transformed locally into the hyperplane $y_n = 0$, while the part of the domain adjacent to $\partial\Omega$ is transformed into $y_n > 0$. In addition, it is assumed that under these changes of coordinates the normals to $\partial\Omega$ are transformed into normals to $y_n = 0$.

Let $\{V_{\nu_2}\}$ be such a finite system of open sets that

$$\Omega \setminus \bigcup_{\nu_1} V_{\nu_1} \subset \bigcup_{\nu_2} V_{\nu_2},$$

and, moreover,

$$\overline{\bigcup_{\nu_2} V_{\nu_2}} \cap \partial\Omega = \emptyset.$$

The space $H_\gamma^l(\Omega)$ is the set of functions defined in $\Omega$ for which the norm

$$\|u\|_{H_\gamma^l(\Omega)} = \sum_{\nu_1} \|\varphi_{\nu_1} u\|_{H_\gamma^l(R_n^+)} + \sum_{\nu_2} \|\varphi_{\nu_2} u\|_{H^l}, \tag{14}$$

is finite, where the norm $\|\varphi_{\nu_1} u\|_{H_\gamma^l}$ is computed in the local coordinates $y^{(\nu_1)}$ as the norm of a function defined in the corresponding half-space, and $\|\varphi_{\nu_2} u\|_{H^l}$ is the usual norm in the Sobolev space.

Let the operator $L$ inside the domain $\Omega$ have the form

$$L(x, D_x) = \sum_{|\alpha| \le r} a_\alpha(x) D_x^\alpha, \tag{15}$$

where the coefficients $a_\alpha$ are sufficiently smooth functions inside the domain $\Omega$. We now define the operator $L$ near the boundary $\partial\Omega$ of the domain $\Omega$, i.e. at points

$$x \in \Omega \cap \left( \bigcup_{\nu_1} V_{\nu_1} \right).$$

We require of the operator $L$ that, for any $\nu_1$, the operator $\widetilde{L}_{\nu_1}$ be representable in the following form:

$$\widetilde{L}_{\nu_1} = \sum_{|\alpha'|+2\alpha_n \leq r} a_{\alpha',\alpha_n}(y) D_y^{\alpha'} B_{y_n}^{\alpha_n}, \tag{16}$$

where $a_{\alpha',\alpha_n}$ are sufficiently smooth functions defined in $\overline{R_n^+} \cap \widetilde{V}_{\nu_1}$, and $\widetilde{V}_{\nu_1}$ is the image of the set $V_{\nu_1}$ under the transformation $x \to y^{(\nu_1)}$. We impose the following restriction on the operator $L$. At every point $x \in \Omega$ the operator $L$ is elliptic, i.e.

$$\sum_{|\alpha|=r} a_\alpha(x) \xi^\alpha \neq 0 \quad \text{for } |\xi| \neq 0, \quad \xi \in R_n, \quad x \in \Omega. \tag{17}$$

For any $\nu_1$, the operator $\widetilde{L}_{\nu_1}$ satisfies, for $|\xi| \neq 0$, $\xi \in R_n$, $y \in \widetilde{V}_{\nu_1} \cap \{y_n \geq 0\}$, the condition

$$\sum_{|\alpha'|+2\alpha_n = r} \tilde{a}_{\alpha',\alpha_n}(y) \xi^{\alpha'} \xi_n^{2\alpha_n} \neq 0. \tag{18}$$

An operator $L$ satisfying these conditions will be called a $B$-elliptic operator in $\Omega$.

**Theorem 4.** *Let the operator $L$ be $B$-elliptic in the domain $\Omega$. Suppose its coefficients satisfy the conditions of Theorem 2. Then, for $\gamma > 0$, the estimate*

$$\|u\|_{H_\gamma^s(\Omega)} \leq C \left( \|Lu\|_{H_\gamma^{s-2m}(\Omega)} + \|u\|_{H_\gamma^0(\Omega)} \right) \tag{19}$$

*holds, and there exists a bounded operator $R$, acting from $H_\gamma^{s-2m}(\Omega)$ into $H_\gamma^s(\Omega)$, such that*

$$LR = I + T, \tag{20}$$

*where $I$ is the identity operator and $T$ is a smoothing operator.*

An analogous assertion holds for the operator $\bar{L}$, defined inside the domain $\Omega$ by means of the operator (15), and near the boundary by means of an operator of the form

$$\widetilde{L}_{\nu_1} = \sum_{|\alpha'|+2\alpha_n=r} \tilde{a}_{\alpha',\alpha_n}(y) D_y^{\alpha'} D_y^{2\alpha_n}, \tag{21}$$

whose coefficients satisfy, near the boundary, the conditions of Theorem 3. The operators $\bar{L}$ and $\widetilde{L}_{\nu_1}$ themselves satisfy conditions (17) and (18), respectively.

Using the Fourier–Bessel transform, one can show that the smoothing operators $T$ thus obtained are completely continuous for a bounded domain $\Omega$. Hence the operators $L$ and $\bar{L}$, acting respectively from $H_\gamma^{2m+k}$ into $H_\gamma^k$, are Noetherian.

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*Note: Figure translations are in progress. See original paper for figures.*

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