CONFORMAL INVARIANCE AND HUYGENS' PRINCIPLE

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Abstract

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CONFORMAL INVARIANCE AND HUYGENS' PRINCIPLE

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I. One of the remarkable properties of the wave equation

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0 \tag{1}$$

is that Huygens' principle holds for it. This means the following: for equation (1), the solution of the Cauchy problem at the point $\mathbf{x} = (x, y, z, t)$ depends only on the values of the initial data in an arbitrarily small neighborhood of the intersection of the characteristic conoid (with vertex at the point $\mathbf{x}$) with the surface carrying the initial data. Hadamard ([1–3]) posed the problem of describing the entire class of linear hyperbolic equations of second order

$$g^{ij}(\mathbf{x})u_{ij} + b^i(\mathbf{x})u_i + c(\mathbf{x})u = 0, \tag{2}$$

for which Huygens' principle is valid (see also papers ([4–8])). We shall consider the case of only four independent variables x^i $(i = 1, \dots, 4)$; therefore below it is assumed that in (2) summation over the indices i, j is carried out from 1 to 4.

Huygens' principle is invariant under transformations (equivalence transformations): a) a nonsingular change of coordinates $x'^i = x'^i(x)$; b) a linear change of the function $u' = \lambda(x)u$, $\lambda(x) \neq 0$; c) multiplication of equation (2) by a function $\nu(x) \neq 0$.

Therefore two equations of the form (2) that can be obtained from one another by the indicated transformations will be regarded as equivalent.

M. Mathisson proved ([9]) that any equation (2) with constant coefficients g^{ij} $(i, j = 1, \dots, 4)$ for which Huygens' principle holds is equivalent to the wave equation (for a number of variables greater than four this is false ([10])). Mathisson' s result seemed to indicate the validity of Hadamard' s conjecture that, for an equation of the form (2), Huygens' principle is true only when this equation is equivalent to the wave equation (Hadamard' s hypothesis). However, recently we gave an example* of an equation of the form (2) for which Huygens' principle holds and which is not equivalent to the wave equation ([11]).

This example was considered in connection with the study of equations (2) with "good" group properties ([12]). It turns out that the group properties of equations (2) and the presence of Huygens' principle are closely connected with one another. Namely, in the case where there exists a Riemannian space with metric tensor $g_{ij}(x)$ and a nontrivial (see below) conformal group for equation (2), Huygens' principle holds if and only if this equation is conformally invariant. In the present paper the main points of the proof of this fact are presented. An explicit formula is given for the solution of the Cauchy problem for an arbitrary conformally invariant equation of the form (2) with a nontrivial conformal group. Among the latter are, naturally, both equation (1) and our example given in ([11]).

* After writing the present paper, I learned that an analogous example had earlier been given by P. Günther

II. A Lie group G is called a conformal group of a Riemannian space V_4 with metric tensor g_{ij} $(i, j = 1, \dots, 4)$, if for any one-parameter subgroup of the group G with infinitesimal operator $X = \xi^i(x)\partial/\partial x^i$ the Killing equations [(13)] are satisfied

$$\xi_{i,j} + \xi_{j,i} = \mu(x) g_{ij} \quad (i, j = 1, \dots, 4). \tag{3}$$

Here $\xi_i = g_{ij}\xi^j$, and the indices after the comma denote covariant differentiation. If for all one-parameter subgroups $\mu(x) \equiv 0$, then the group G is called a group of motions. The conformal group of the space V_4 is called **trivial** if it is a group of motions in some Riemannian space conformal to the space V_4. A conformal group that is not trivial is called a **nontrivial** conformal group [(14)]. For simplicity we restrict ourselves to the case of analytic functions $g_{ij}(x)$.

Below we shall use the following fact, which follows from the results of P. F. Bilyarov (see, for example, [(14)], Ch. VII).

Lemma 1. *Any space V_4 of signature $(---+)$ with a nontrivial conformal group can, by a change of coordinates and a transition to a conformal space, be brought to a space with tensor g^{ij} $(g_{ij}g^{jk} \neq \delta_i^k)$ of the form*

$$g^{ij}(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -f(x^1 - x^4) & -h(x^1 - x^4) & 0 \\ 0 & -h(x^1 - x^4) & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad f - h^2 > 0. \tag{4}$$

In this case the order of the conformal group is equal to either 6, or 7, or 15.

III. Let us write equation (2) in the form

$$L(u) \equiv g^{ij} u_{,ij} + a^i u_{,i} + cu = 0, \tag{5}$$

using covariant derivatives in the Riemannian space V_4 with metric tensor g_{ij}. The invariance properties of equation (5) with respect to continuous groups of transformations are described [15] by means of the functions $K_{ij} = a_{i,j} - a_{j,i}$ $(i, j = 1, \dots, 4)$ and $H = -2c + a^i{}_{,i} + \frac{1}{2} a^i a_i + \frac{1}{3} R$, where R is the scalar curvature of the space V_4. Namely, the coordinates $\xi^i(x)$ of the infinitesimal operator of any one-parameter subgroup of the Lie group admitted by equation (5) are determined as solutions of equations (3) and of the equations

$$\xi^k H_{,k} + \mu H = 0, \qquad (K_{il}\xi^l)_{,j} - (K_{jl}\xi^l)_{,i} = 0 \quad (i, j = 1, \dots, 4). \tag{6}$$

Therefore the group admitted by equation (5) is a subgroup of the group of conformal transformations of V_4.

For what follows it is important to determine which equations (5) are invariant with respect to the entire conformal group in V_4 (we shall call such equations **conformally invariant**). For spaces with a nontrivial conformal group, this question is answered by the following

Theorem 1. *In any space V_4 of signature $(---+)$ with a nontrivial conformal group, every conformally invariant equation of the form (5) is equivalent to the equation*

$$g^{ij} u_{,ij} + \frac{1}{6} R u = 0. \tag{7}$$

Proof. By virtue of Lemma 1, it is sufficient to prove this theorem for spaces with tensor $g^{ij}(x)$ of the form (4), for which the conformal group is easily computed [14]. Solving equations (6), we obtain

$$K_{ij} = 0 \quad (i, j = 1, \dots, 4), \qquad H = 0, \tag{8}$$

from which the assertion of the theorem follows (see, for example, [3,15]).

It is interesting to note that for spaces with a trivial conformal group, besides equation (7), there exists at least one more conformally invariant equation not equivalent to equation (7).

IV. Consider, for the equation

$$u_{tt} - u_{xx} - f(x - t) u_{yy} - 2h(x - t) u_{yz} - u_{zz} = 0, \tag{9}$$

equivalent to equation (7) in a space with a tensor g^{ij} of the form (4), the Cauchy problem with initial data

$$u\big|_{t=0} = 0, \tag{9$_1$}$$

$$u_t\big|_{t=0} = \varphi(x, y, z). \tag{9$_2$}$$

Here $x = x^1,\ y = x^2,\ z = x^3,\ t = x^4$. The restriction (9_1) is immaterial, since for equation (9) the Cauchy problem with arbitrary initial data can be reduced to problem (9)–(9_2), and it does not affect the fact of the presence or absence of Huygens' principle.

The solution of problem (9)–(9_2) has the form

$$u(x, y, z, t) = \frac{1}{4\pi}\int_{x-t}^{x+t} d\xi \int_0^{2\pi} \varphi(\xi,\ y + A\cos\theta,\ z + B\cos\theta + C\sin\theta)\, d\theta, \tag{10}$$

where

$$A = \{(x + t - \xi)[F(\xi) - F(x - t)]\}^{1/2},$$

$$B = [H(\xi) - H(x - t)]\left[\frac{x + t - \xi}{F(\xi) - F(x - t)}\right]^{1/2},$$

$$C = \left\{(x + t - \xi)\left[\xi - x + t - \frac{(H(\xi) - H(x - t))^2}{F(\xi) - F(x - t)}\right]\right\}^{1/2},$$

and F and H are antiderivatives of the functions f and h, respectively.

From formula (10) it is clear that (for arbitrary $f, h \in C^1(R)$, $f > h^2$) Huygens' principle is valid for equation (9). Indeed, the solution of problem (9)–(9_2) at the point $\mathbf{x}$ depends only on the values of the function φ on the intersection of the hyperplane $\tau = 0$ with the characteristic conoid with vertex at the point $\mathbf{x}$, which is given by the equation $\Gamma(\mathbf{x}, \xi) = 0$. Here $\Gamma(\mathbf{x}, \xi)$ denotes the square of the geodesic distance between the points $\mathbf{x} \equiv (x, y, z, t)$ and $\xi \equiv (\xi, \eta, \zeta, \tau)$, and for equation (9) has the form

$$\Gamma(\mathbf{x}, \xi) = (t - \tau)^2 - (x - \xi)^2 -$$

$$-\frac{x - t - \xi + \tau}{(x - t - \xi + \tau)(F(x - t) - F(\xi - \tau)) - (H(x - t) - H(\xi - \tau))^2}\times$$

$$\times\{(x - t - \xi + \tau)(y - \eta)^2 - 2(H(x - t) - H(\xi - \tau))(y - \eta)(z - \zeta) +$$

$$+(F(x-t)-F(\xi-\tau))(z-\zeta)^2\}. \tag{11}$$

Our example in ([11]) corresponds to the case $h\equiv 0$. If one sets $h\equiv 0$, $f\equiv 1$, then from (10) one can obtain the well-known Poisson formula for the wave equation (1).

V. **Theorem 2.** *Let a Riemannian space V_4 of signature $(---+)$ have a nontrivial conformal group. Then, for equation (5), Huygens' principle is valid if and only if this equation is conformally invariant, i.e. equivalent to equation (7).*

We note that this theorem contains a result of M. Mathisson ([9]). The proof of Theorem 2 can be obtained by the method proposed by Hadamard ([3]) for the case $g^{ij}=\text{const}$. By virtue of Lemma 1 and Theorem 1, it suffices for us to prove that from the validity of Huygens' principle for equation (5) with leading coefficients of the form (4) there follows the fulfillment

of equations (8). We shall establish the fulfillment of (8) for the equation adjoint to equation (5), whence, by virtue of the self-adjointness of equation (7), the validity of the theorem will follow.

Hadamard showed that, for the equation adjoint to equation (5), Huygens' principle is valid if and only if the equation $L(W_0)=0$ is satisfied along the characteristic conoid with vertex at an arbitrary point $\mathbf{x}_0$. Here

$$W_0=\exp\left\{-\frac{1}{4}\int_{\mathbf{x}0}^{\mathbf{x}}(L(\Gamma)-c\Gamma-8)\frac{dS}{S}\right\},$$

the integral is taken along the geodesic joining the points $\mathbf{x}=(x,y,z,t)$ and $\mathbf{x}_0=(x_0,y_0,z_0,t)$; $\Gamma=S^2$ is the square of the geodesic distance. In this, W_0 is regarded as a function of the point $\mathbf{x}$.

Applying Hadamard' s criterion, written in the form $L(W_0)=\lambda\Gamma$ with an undetermined (regular) coefficient $\lambda=\lambda(x)$, we obtain that at the point x_0 equations (8) are satisfied. By virtue of the arbitrariness of the point x_0, the validity of the theorem follows.

VI. In paper [12], equation (7) was considered as an equation describing the propagation of light waves in the Riemannian space V_4, proceeding from the fact of the conformal invariance of this equation. Even earlier (and, apparently, for the first time) equation (7) was considered in paper [16] also from the point of view of conformal invariance. Theorem 2 gives a certain physical justification for the choice of equation (7) as the wave equation in a Riemannian space (possibly up to some equivalence transformation b), c) for $\nu=1/\lambda$). Namely, considering the radiation problem (5) and using Theorem 2, we see that, in the case of the existence of a nontrivial conformal group in the space V_4, sharp light signals are transmitted and can be received as sharp, if the propagation of light waves in the space V_4 is described by equation (7).

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Note: Figure translations are in progress. See original paper for figures.