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IDEALLY OBSERVABLE SYSTEMS

MATHEMATICS

1970

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Abstract

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UDC 517.91

MATHEMATICS

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IDEALLY OBSERVABLE SYSTEMS

(Presented by Academician L. S. Pontryagin on 13 X 1969)

§ 1. In the n-dimensional Euclidean space R^n, the motion of a vector x takes place, satisfying the differential equation

$$\dot{x} = Ax + Bu, \tag{1}$$

where A is a square matrix of size $n \times n$, u is a control vector of dimension $p \geqslant 1$ from the Euclidean space R^p, and B is a rectangular matrix of size $p \times n$. It is assumed that the control $u = u(t)$ is a measurable function, bounded in absolute value, on the interval $[0, \Delta]$, where $\Delta > 0$.

There is an observer to whom the following are known:

a) the dynamical capabilities of the object x—the matrices A, B;

b) a rectangular matrix G of size $n \times q$ of nonzero rank and the vector function

$$y(t) = Gx(t), \quad 0 \leqslant t \leqslant \Delta. \tag{2}$$

The observer' s problem is to reconstruct uniquely the vector $x(0) = x_0$ from the function $y(\cdot)$.*

Systems (1), (2) for which the observer can uniquely reconstruct the vector $x(0) = x_0$ from any possible function $y(\cdot)$ will be called **ideally observable**. The remaining systems will be called **non-ideally observable**.

In Kalman' s observability theory (see (1, 2)) it is assumed that, in addition to the information indicated in items a), b), the observer also knows the control $u(\cdot)$ of the object x. It is not difficult to see that from the ideal observability of system (1), (2) there follows its complete observability in the sense of Kalman (see (1)). The converse, generally speaking, is false (see § 6, Example 1).

§ 2. In the paper an algorithm is constructed that makes it possible to determine, from the matrices A, B, G, whether system (1), (2) is ideally observable or not.

This algorithm consists in computing a finite number (not more than $p + 1$) of ranks of constant matrices and solving a finite number (not more than p) of

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systems of linear algebraic equations. From it, in particular, it follows that the property of system (1), (2) of being ideally observable does not depend on the length of the observation interval $\Delta > 0$.

Theorem 1. *System (1), (2) is ideally observable if and only if the system* $\dot{x} = (A + BM)x,\ y = Gx$ *(where M is an arbitrary matrix of size $n \times p$) is completely observable in the sense of Kalman (see (1)). System (1), (2) is non-ideally observable if and only if there exists a matrix M_0 of size $n \times p$ such that the system* $\dot{x} = (A + BM_0)x,\ y = Gx$ *is not completely observable.*

§ 3. Suppose that system (1), (2) is ideally observable; then on the functions $y(t) = Gx(t)$, $0 \leqslant t \leqslant \Delta$, where $x(t)$ is a solution of equation (1), an operator F is defined, possessing the property $Fy(\cdot) = x_0$. It is not difficult to see that the ope-

* The dot in parentheses here and below means that the function is considered not at a point, but on the whole observation interval $[0, \Delta]$.

operator F has the property of linearity, i.e. $F[\lambda_1 y_1(\cdot) + \lambda_2 y_2(\cdot)] = \lambda_1 F y_1(\cdot) + \lambda_2 F y_2(\cdot)$, where λ_1, λ_2 are arbitrary numbers.

Denote by the symbol $\mathfrak{M}_P$ the collection of functions $y(\cdot)$ having the property: one of the admissible controls $u(\cdot)$ generating the function $y(\cdot)$ satisfies the condition

$$u(t) \in P, \qquad 0 \le t \le \Delta, \tag{3}$$

where P is a convex compact set in R^p. Denote by F_P the restriction of the operator F to $\mathfrak{M}_P$. Put $\|y(\cdot)\|_C = \max_{0 \le t \le \Delta} |y(t)|$, where $|y(t)|$ denotes the modulus of the q-dimensional vector $y(t)$.

Theorem 2. *The operator F_P is uniformly continuous on $\mathfrak{M}_P$ in the metric of the space $C[0, \Delta]$, i.e. for every $\varepsilon > 0$ one can choose a $\delta(\varepsilon) > 0$ such that, for any $y_1(\cdot), y_2(\cdot)$ from $\mathfrak{M}_P$ satisfying the condition $\|y_1(\cdot) - y_2(\cdot)\|_C < \delta(\varepsilon)$, the inequality $|F_{Py} 1(\cdot) - F_{Py} 2(\cdot)| < \varepsilon$ holds.*

Put

$$\|y(\cdot)\|_{L_2} = \left(\int_0^\Delta |y(s)|^2 \, ds \right)^{1/2}.$$

Theorem 3. *The operator F_P is uniformly continuous on $\mathfrak{M}_P$ in the metric of the space $L_2[0, \Delta]$, i.e. for every $\varepsilon > 0$ one can choose a $\delta(\varepsilon) > 0$ such that, for any $y_1(\cdot), y_2(\cdot)$ from $\mathfrak{M}_P$ satisfying the condition $\|y_1(\cdot) - y_2(\cdot)\|_{L_2} < \delta(\varepsilon)$, the inequality*

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$$|F_{Py}1(\cdot) - F_{Py}2(\cdot)| < \varepsilon.$$

Theorems 2, 3 give important information on the properties of the operator F and are of interest for practical applications.

§ 4. Let system (1), (2) be ideally observable. The problem arises of effectively computing the operator F.

We shall assume that the observer has additional information: one of the admissible controls $u(\cdot)$ generating the function $y(\cdot)$ satisfies condition (3), and the convex compact set P is known to the observer.

Such a formulation of the observation problem is natural for differential pursuit games with incomplete information, in which the pursuer (who is also the observer) is informed about the dynamics of the evader, i.e. knows the matrices A, B and the set P, but, with respect to the vector $x(t)$, knows only its "projection" $y = Gx(t)$.

To solve the posed problem, an auxiliary problem is considered: that of minimizing the functional

$$I(u(\cdot)) = \int_0^\Delta |w(t) - Gx(t)|^2 \, dt, \tag{4}$$

(where $w(\cdot)$ is an arbitrary q-dimensional vector function from $L_2[0,\Delta]$) on the free trajectories of equation (1), with $u \in P$. It is proved that there exists a minimum of the functional (4) and that it is attained on a unique trajectory $\tilde{x}(\cdot)$ of equation (1). Thus, on the elements $w(\cdot) \in L_2[0,\Delta]$ there is defined, generally speaking, a nonlinear operator $F'_P : F'_P w(\cdot) = \tilde{x}(\cdot)$. Introduce also the operator F^2, which is defined on trajectories of equation (1) by the equality $F^2 x(\cdot) = x(0)$. Now define on the elements $w(\cdot) \in L_2[0,\Delta]$ the operator F^3_P: $F^3_P w(\cdot) = F^2 F'_P w(\cdot) = \tilde{x}(0)$.

It is not difficult to see that if $w(\cdot) = y(\cdot) \in \mathfrak{M}_P$, then $F^3_P w(\cdot) = x(0) = x_0$. Thus, on $\mathfrak{M}_P$ the operator F^3_P coincides with F_P, and it is important to learn to compute the operator F^3_P effectively for arbitrary $w(\cdot) \in L_2[0,\Delta]$.

First consider the case when $w(\cdot)$ is continuously differentiable on $[0,\Delta]$. It is proved that the maximum principle of L. S. Pont-

Pontryagin (see (3)) is a necessary and sufficient condition in the problem of minimizing the functional (4) on the free trajectories of equation (1). Using this fact, it is proved that the method of successive approximations set forth in § 4 of [4] can be applied for the effective computation of $\tilde{x}(\cdot) = F_{P'} w(\cdot)$. Thus, in the case of continuous differentiability of the function $w(\cdot)$, we have an effective method for computing $F_{P'} w(\cdot)$, and hence also for $F_{P^3} w(\cdot)$.

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To compute $F_{P^3}w(\cdot)$ in the case of an arbitrary $w(\cdot) \in L_2[0,\Delta]$, one may proceed as follows.

Consider a sequence of continuously differentiable functions $w_n(\cdot)$ such that $\|w_n(\cdot) - w(\cdot)\|_{L_2} \to 0$ as $n \to +\infty$.

Theorem 4. *The operator F_{P^3} is continuous at every element $\hat{w}(\cdot) \in L_2[0,\Delta]$: for a given $\varepsilon > 0$ one can find a $\delta(\varepsilon) > 0$ such that, for any $w(\cdot) \in L_2[0,\Delta]$ satisfying the inequality $\|w(\cdot) - \hat{w}(\cdot)\|_{L_2} < \delta(\varepsilon)$, the inequality $|F_{P^3}w(\cdot) - F_{P^3}\hat{w}(\cdot)| < \varepsilon$ holds.*

By Theorem 4, the sequence of vectors $\tilde{x}_n(0) = F_{P^3}w_n(\cdot)$ converges to the vector $\tilde{x}(0) = F_{P^3}w(\cdot)$.

§ 5. In this paragraph we shall consider the problem of observing system (1) by means of the function

$$v(t) = Gx(t) + \omega(t), \qquad 0 \le t \le \Delta. \tag{5}$$

The function $\omega(\cdot)$ is assumed to be unknown to the observer (it is called the disturbance); he knows only that $\omega(\cdot) \in L_2[0,\Delta]$ and that $\|\omega(\cdot)\|_{L_2} \le \mu$.

We shall consider the problem under the assumption that the observer has the additional information mentioned at the beginning of § 4. The system (1), (2) is assumed to be ideally observable.

Put $\tilde{x}(\cdot) = F_{P^3}v(\cdot)$. It is easy to see that $\|v(\cdot) - G\tilde{x}(\cdot)\|_{L_2} \le \mu$, whence, and from formula (5), it follows that

$$\|Gx(\cdot) - G\tilde{x}(\cdot)\|_{L_2} \le 2\mu. \tag{6}$$

From this inequality and from Theorem 3 it follows that $|\tilde{x}(0) - x(0)| \to 0$ as $\mu \to 0$. For a more precise study of the character of the dependence on μ of the degree of approximation of the vector $\tilde{x}(0)$ to the vector $x(0)$, we introduce the scalar function $g(r)$. To this end, consider the family of functions $z(t,x)$ of the form

$$z(t,x) = Ge^{tA}x + \int_0^t Ge^{(t-s)A}Bu(s)\,ds, \qquad 0 \le t \le \Delta,$$

where $u(s) \in Q = P - P$ (the sign $-$ denotes the algebraic difference of sets). The set Q, as is not difficult to see, is a convex compact set. Put

$$f(x) = \min_{u(\cdot)} \|z(\cdot,x)\|_{L_2}, \qquad g(r) = \min_{|x|=r} f(x).$$

It is proved that $g(r)$ is a continuous function of r, with $g(0) = 0$ and $g(r) > 0$ for $r > 0$. It is proved that $g(r)$ increases monotonically together with r.

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Therefore there exists a continuous inverse function $r = r(g)$ satisfying the condition $g(r(g_0)) = g_0$. From inequality (6) we conclude that the vector $\tilde{x}(0)$ approximates the vector $x(0)$ with accuracy up to $\varepsilon(\mu) = r(2\mu)$, where $\varepsilon(\mu) \to 0$ as $\mu \to 0$. Note that the function $f(x)$ is effectively computable by means of L. S. Pontryagin' s maximum principle (see (3)), so that the function

$$g(r) = \min_{|x|=r} f(x)$$

is effectively computable.

§ 6. **Examples.**

Example 1.

$$\dot{x}_1 = x_2 + u_1, \qquad \dot{x}_2 = u_2, \qquad y_1 = x_1.$$

This system is not ideally observable, although it is completely observable in the sense of Kalman (see [1]).

Example 2.

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u, \qquad y_1 = x_1.$$

This system is perfectly observable.

Example 3.

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\alpha x_2 + u, \qquad y_1 = x_1,$$

where $\alpha = \text{const}$. This system is perfectly observable.

In conclusion, I wish to note that the formulation of the problem of perfect observation belongs to L. S. Pontryagin. I express my deep gratitude to L. S. Pontryagin and E. F. Mishchenko for their constant attention to my work.

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Received
24 IX 1969

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Note: Figure translations are in progress. See original paper for figures.

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