ON THE APPROXIMATE SOLUTION OF OPERATOR EQUATIONS OF THE FIRST KIND

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Abstract

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Consider the linear operator equation

$$Ax = f, \qquad x \in X, \qquad f \in R(A) \subset X, \tag{1}$$

where X is a Banach space, $A[X \to X]$ is a linear closed operator, and continuous dependence of x on f does not hold.

When solving equation (1), usually instead of the operator A one considers some h-parametric family of operators A_h, $h = (h_1, h_2, \dots, \dots, h_n)$, each of which is, in a certain sense, close to the original one:

$$\|(A - A_h)x\| \le \alpha(h), \qquad x \in D \subset X,$$

where $\alpha(h)$ is a sufficiently small scalar function of the vector variable h, and D is some subset of the space X to which the solution of equation (1) belongs.

The right-hand side of equation (1) under real conditions is given with error. Consequently, one may assume that instead of the function f a function f_h is known such that $\|f - f_h\| \le \beta(h)$, where $\beta(h)$ has the same meaning as $\alpha(h)$.

Therefore, along with the original equation (1), consider the "approximate" equation

$$A_{hx}h = f_h, \qquad x_h \in X, \qquad f_h \in R(A_h) \subset X, \tag{2}$$

where the linear closed operator $A_h[X \to X]$, for $h \ne h^0$, $h^0 = (h_1^0, h_2^0, \dots, \dots, h_n^0)$, has a bounded inverse. With respect to the solution x_h of equation (2), we shall assume that as $h \to h^0$ it tends to the solution x of equation (1). We shall not dwell on the question of sufficient conditions for the convergence of x_h to x as $h \to h^0$. These conditions have been obtained in a number of works ([1−3] and others).

Here it should be noted that, for the convergence of x_h to x as $h \to h^0$, generally speaking, it is necessary to require that the condition

$$\lim_{h\to h^0}\alpha(h)=\lim_{h\to h^0}\beta(h)=0$$

be satisfied.

For the approximate solution of the operator equation (2), iterative methods are usually used. However, if h is close to h^0, then the solution of equation (2) behaves unstably with respect to variations of the operator A_h and of the right-hand side f_h.

Therefore, when solving equation (2), it is necessary to take into account the errors arising as a result of replacing the operator A by the operator A_h and, correspondingly, f by f_h. The choice of the number of iterations as a regularization parameter was used in works ([2,4] and others).

However, this method of iterative regularization for a general operator equation of the first kind is essentially not constructive, since for its use it is necessary to know an expression for the modulus of continuity of the inverse operator.

In the present work a constructive expression is given for the number of iterations in terms of ordinarily known quantities.

We shall solve equation (2) by means of the universal iterative process ([5,7])

$$x_h^{k+1}=x_h^k+B_h(f_h-A_{hx}h^k),\qquad k=0,1,2,\dots,\tag{3}$$

$$x_h^0=B_{hf}h,\qquad \|f_h\|\neq 0,$$

where the operator $B_h[X\to X]$ satisfies the condition

$$0<\|E-B_{hA}h\|=q<1\qquad \text{for } h\neq h^0.$$

Obviously,

$$x_h-x=A_h^{-1}[(f_h-f)+(A-A_h)x],$$

therefore

$$\|x_h-x\|\leq\|A_h^{-1}\|\|f-f_h\|+\|(A-A_h)x\|.\tag{4}$$

Next, from (3) it follows that

$$x_h-x_h^{k+1}=[E-B_{hA}h]^{k+2}A_h^{-1}f_h,$$

$$\|x_h - x_h^{k+1}\| \leq q^{k+2}\|A_h^{-1}\|\,\|f_h\|. \tag{5}$$

Taking (4) and (5) into account, we obtain

$$\|x-x_h^{k+1}\| \leq \|x_h^{k+1}-x_h\|+\|x_h-x\| \leq q^{k+2}\|A_h^{-1}\|\,\|f_h\|+\|A_h^{-1}\|(\alpha(h)+\beta(h)). \tag{6}$$

It follows from estimate (6) that, as $k \to \infty$, the first term on the right-hand side of (6) tends to zero, while the second does not depend on k.

Therefore it is meaningful to terminate the iterative process (3) at $k = k_0 = [\delta]^+$, where δ is the root of the equation

$$q^{\delta+2}\|A_h^{-1}\|\,\|f_h\| = \|A_h^{-1}\|(\alpha(h)+\beta(h)), \tag{7}$$

$$[\delta]^+ = \begin{cases} [\delta], & \text{if } [\delta] > 0, \\ 0, & \text{if } [\delta] \leq 0, \end{cases}$$

$[\delta]$ is the integer part of δ.

From equality (7) we obtain

$$\delta = \frac{1}{\ln q} \ln \frac{\alpha(h)+\beta(h)}{\|f_h\|} - 2. \tag{8}$$

We shall show that

$$\lim_{h\to h^0} x_h^{k_0+1} = x.$$

Indeed, according to (5) we have

$$\|x_h^{k_0+1} - x\| \leq \|x_h^{k_0+1} - x_h\| + \|x_h - x\| \leq q^{k_0+2}\|x_h\| + \|x_h - x\|$$

$$\leq q^{k_0+2}(\|x_h - x\| + \|x\|) + \|x_h - x\|.$$

Next,

$$q^{k_0+2} = q^{\delta-\{\delta\}+2} = q^{-\{\delta\}}\frac{1}{\|f_h\|}(\alpha(h)+\beta(h)), \qquad 0 \leq \{\delta\} < 1,$$

where $\{\delta\}$ is the fractional part of the number δ.

Therefore we finally obtain

$$\|x_h^{k_0+1} - x\| \leq q^{-\{\delta\}} \frac{\alpha(h) + \beta(h)}{\|f_h\|} (\|x_h - x\| + \|x\|) + \|x_h - x\|,$$

from which our assertion follows.

Thus, the following is valid.

Theorem. Let X be a Banach space, and let $A[X \to X]$ be a linear closed operator. Consider the two equations (1) $Ax = f$, $f \in R(A)$, and (2) $A_{hx}h = f_h$, $h = (h_1, h_2, \dots, h_n)$, $x_h \in X$, $f_h \in R(A_h) \subset X$, where continuous dependence of x on f does not take place. Suppose further that $\|f - f_h\| \leq \beta(h)$, $\|(A - A_h)x\| \leq \alpha(h)$, $x \in D$, $\lim_{h \to h^0} \alpha(h) = \lim_{h \to h^0} \beta(h) = 0$, $h^0 = (h_1^0, \dots, h_n^0)$, and D is some subset of the space X to which the solution of equation (1) belongs. Then, if the linear closed ope-

operator $A_h[X \to X]$ for $h \neq h^0$ has a bounded inverse and the conditions are satisfied: 1) $\lim_{h \to h^0} x_h = x$, 2) the operator $B_h[X \to X]$ satisfies the condition $0 < \|E - B_{hA}h\| = q < 1$ for $h \neq h^0$, then $x = \lim_{h \to h^0} x_h^{k_0+1}$, where

$$x_h^{k+1} = x_h^k + B_h(f_h - A_{hx}h^k), \quad k = 0, 1, \dots,$$

$$x_h^0 = B_{hf}h, \quad k_0 = [\delta]^+, \quad \delta = \frac{1}{\ln q} \ln \frac{\alpha(h) + \beta(h)}{\|f_h\|} - 2.$$

Let us note that the expression for the number of iterations k_0 is constructive, since this expression contains the usually known quantities $q, \alpha(h), \beta(h), \|f_h\|$.

Obviously, everything said above is also valid for the case when the operator A has a bounded inverse.

For solving the operator equation (2), a nonstationary universal iterative process is also often used,

$$x_h^{k+1} = x_L^k + B_k(h)(f_h - A_{hx}h^k), \quad k = 0, 1, 2, \dots, \quad x_h^0 = B_0(h) f_h, \tag{9}$$

where $B_k(h)[X \to X]$ is an operator satisfying the condition

$$0 < \|E - B_k(h) A_h\| = q_k \leq q < 1$$

for $h \neq h^0$.

The condition $q_k \leq q$ is natural, since in many iterative processes the quantities q_k rather quickly reach their asymptote.

It is not difficult to verify that in this case the estimate

$$\|x_h - x_h^{k+1}\| \leq q_{kq_{k-1}} \cdots q_1 q_0^2 \|A_h^{-1}\| \cdot \|f_h\| \leq q^{k+2} \|A_h^{-1}\| \cdot \|f_h\| \tag{10}$$

is valid.

Since the last estimate coincides with estimate (5), when solving the operator equation (1) by means of the universal nonstationary process (9) the theorem stated above holds.

In the case when A_h is a matrix of order n, when solving equation (1) by means of the universal process (3), the accumulation of rounding errors may be taken into account.

For this we use the known estimates of the rounding error of the iterative process (3) when it is implemented on a computer with fixed binary precision t (\^{}6). Let $2^{-t}\delta_k$ be the error accumulated over k steps, and let $\tilde{x}_h^k$ be the k-th approximation computed on the machine. Then the following majorizing estimate holds:

$$\|\delta_k\|_2 \leq \beta n^{3/2} 2^{-t-1}, \quad \beta = (1-q^k)/(1-q), \quad 2^{-t}\delta_k = x_h^k - \tilde{x}_h^k.$$

Further, we have

$$\|x - \tilde{x}_h^{k+1}\| \leq \|x_h - x_h^{k+1}\| + \|x_h^{k+1} - \tilde{x}_h^{k+1}\| + \|x - x_h\|.$$

Using (4), (5), and (10), we rewrite the last inequality as

$$\|x - \tilde{x}_h^{k+1}\| \leq q^{k+2}\|A_h^{-1}\|\|f_h\| + \|A_h^{-1}\|(\alpha(h) + \beta(h)) + \frac{1-q^{k+1}}{1-q} n^{3/2} 2^{-2t-1}.$$

Therefore, instead of (7) in the present case we shall have the following equation:

$$q^{\delta+1}\left(q\|f_h\| - \frac{1}{1-q} n^{3/2} \frac{2^{-2t-1}}{\|A_h^{-1}\|}\right) = \alpha(h) + \beta(h) + \frac{1}{1-q} n^{3/2} \frac{2^{-2t-1}}{\|A_h^{-1}\|}. \tag{11}$$

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Note: Figure translations are in progress. See original paper for figures.