LINEAR MAGNETOHYDRODYNAMIC THEORY OF THE DEVELOPMENT OF INHOMOGENEITIES IN AXISYMMETRIC MODELS OF THE UNIVERSE WITH A COSMOLOGICAL MAGNETIC FIELD

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Abstract

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LINEAR MAGNETOHYDRODYNAMIC THEORY OF THE DEVELOPMENT OF INHOMOGENEITIES IN AXISYMMETRIC MODELS OF THE UNIVERSE WITH A COSMOLOGICAL MAGNETIC FIELD

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Solutions of the Einstein equations for a gravitating ideal gas that admit a three-parameter group of motions have recently been intensively studied in connection with the problem of choosing a general-relativistic model of a homogeneous universe. The accumulated body of observational data does not make it possible to make a categorical choice of a cosmological model. Hoyle ([1]) put forward the idea of the possible existence of a cosmological magnetic field.

Let the group properties of a certain solution of the Einstein-Maxwell equations make it possible to write the interval in the coordinate system comoving with the gas in the form*

$$ds^2 = dt^2 - a^2(t)\left(dr^2 + \frac{\sin^2\sqrt{k}r}{k}\,d\varphi^2\right) - b^2(t)dz^2, \tag{1}$$

where $k = 1, 0, -1$, respectively, for closed, flat, and open models (the magnetic field is directed along the z-axis; its energy density W does not depend on r, φ, z and varies with time according to the law $W = \mathrm{const}/a^4$; the energy density of the gas in the reference frame (1) also depends only on time).

We shall regard one of the solutions (1) as the basic cosmological background and superpose on it arbitrary small perturbations of the distribution and velocity of matter, and of the gravitational and electromagnetic fields. In what follows we shall assume that the matter has ideal electrical conductivity. Then the perturbed electric field will be determined through the 4-velocity and magnetic field from the formula $F_{ik}u^k = 0$. The medium will be described by the model of an ideal gas. In the perturbed Riemannian space with metric tensor $g_{ij} = g_{ij}^{(0)} + h_{ij}$ (where $g_{ij}^{(0)}$ is determined by the interval (1)) we introduce a synchronous reference system. Then $h_{00} = h_{0\alpha} = 0$, since $g^{(0)}_{0\alpha} = 0$, $g^{(0)}_{00} = 1$. From the condition $u^i u_i = 1$ for the 4-velocity it follows that $\delta u_0 = 0$. Eliminating the electric field from the system of Maxwell equations $\nabla_i \delta F_{kl} + \nabla_k \delta F_{li} + \nabla_l \delta F_{ik} = 0$, we obtain the relations

$$\frac{\partial}{\partial t}\delta H^3 = -\nabla_\mu \delta u^\mu; \qquad \frac{\partial}{\partial t}\delta H^\mu = \delta u^\mu. \tag{2}$$

Here and below μ, η, ν run over the values $1, 2$; covariant differentiation is performed with the aid of the metric

$$a^2(t)\left(dr^2 + \frac{\sin^2\sqrt{k}r}{k}\,d\varphi^2\right);$$

the notations

$$\frac{\delta F_{12}}{F^{(0)}_{12}} = \delta H^3; \qquad \frac{\delta F_{23}}{F^{(0)}_{12}} = \frac{\partial}{\partial z}\delta H^1; \qquad \frac{\delta F_{31}}{F^{(0)}_{12}} = \frac{\partial}{\partial z}\delta H^2; \qquad F^{(0)}_{12} —$$

* Here and below a system of units is used in which the speed of light is equal to unity; the properties of the cosmological model (1) are properties of a system of ordinary differential equations for the functions $a(t)$ and $b(t)$.

the only nonzero component of the magnetic field for the basic model (1).

The equations of magnetic-field induction, together with the system of Einstein equations

$$\delta R^i_k - \frac{1}{2}\delta^i_k \delta R = \varkappa \delta T^i_k \tag{3}$$

form a closed system of equations for determining the evolution of inhomogeneities in model (1), if, in accordance with the physics of matter, at a given temperature and density one specifies the equation of state of an ideal gas. The total energy-momentum tensor—ideal gas plus electromagnetic field—on the right-hand side of (3) has the form

$$\delta T^\psi_\eta = -\left[\delta p + W(2\delta H^3 + h_\perp)\right]\delta^\psi_\eta; \qquad \delta T^\psi_3 = 2W\frac{\partial}{\partial z}\delta u^\psi;$$

$$\delta T^3_3 = -\delta p + W(2\delta H^3 + h_\perp); \qquad \delta T^\mu_0 = (p + \varepsilon + 2W)\delta u^\mu;$$

$$\delta T^3_0 = (p+\varepsilon)\delta u^3; \qquad \delta T^0_0 = \delta\varepsilon + W(2\delta H^3 + h_\perp).$$

Here and below $h_\perp = h_1^1 + h_2^2 \equiv h_\mu^\mu$. It can be shown that the system of linear equations (2) and (3) splits into two independent subsystems if the unknowns h_η^μ, h_3^μ, δu^μ, δH^μ are decomposed into "potential" and "vortical" terms:

$$h_\eta^\mu = \frac{1}{2}\delta_\eta^\mu h_\perp + \left(\nabla^\mu \nabla_\eta Q - \frac{1}{2}\delta_\eta^\mu \nabla^\nu \nabla_\nu Q\right) + \nabla^\mu \widetilde{B}_\eta + \nabla_\eta \widetilde{B}^\mu$$

(here, instead of the three unknowns h_η^μ, four unknowns $h_\perp, Q, \widetilde{B}^\mu$ are introduced, with the additional condition $\nabla_1 \widetilde{B}^1 + \nabla_2 \widetilde{B}^2 = 0$); $h_3^\mu = \dfrac{\partial}{\partial z}(\nabla^\mu L + \widetilde{L}^\mu)$, where $\nabla_\mu \widetilde{L}^\mu = 0$; $\delta u_\eta = \nabla_\eta \varphi + \tilde{u}_\eta$, $\nabla_1 \tilde{u}^1 + \nabla_2 \tilde{u}^2 = 0$; $\delta H^\mu = \nabla^\mu H + \widetilde{H}^\mu$, $\nabla_1 \widetilde{H}^1 + \nabla_2 \widetilde{H}^2 = 0$.

The system of equations for "vortical"perturbations $\widetilde{L}^\mu$, $\widetilde{B}^\mu$, $\tilde{u}^\mu$, $\widetilde{H}^\mu$ is separated out from the system of Maxwell-Einstein equations; as a result, the order of the latter in time is lowered by 4. "Vortical"perturbations do not affect the evolution of the matter density; by eliminating $\widetilde{L}^\mu$, $\widetilde{H}^\mu$, $\tilde{u}^\mu$, one can obtain one fourth-order equation in time for $(B^\mu)^\cdot$. It is convenient to study equations (2), (3) in spherical waves in the variables r, φ, and in a Fourier representation along the z-axis (on the expansion of functions in spherical waves and the Fourier integral in Lobachevsky spaces, see ([2])). After the system has been split into "vortical" and "potential" subsystems, the differentiations in each of the subsystems with respect to the spatial coordinates enter only in the form of the operators $\partial^2/\partial z^2$ and $\nabla_\mu \nabla^\mu$; for the modes, in place of these operators one must substitute the multiplication operators by the numbers $-k^2$, $-w^2/a^2$, respectively, where $-w^2$ is the eigenvalue of the Laplace operator for the metric $dr^2 + \dfrac{\sin^2 \sqrt{k}\, r}{k}\, d\varphi^2$. It is easy to show that the equations for "vortical" perturbations with scale much smaller than the "horizon" (i.e., for $k^2/b^2 + w^2/a^2 \equiv \Delta^2 \gg 1/\sqrt{\varkappa(\varepsilon + W)}$) contain the solution for a gravitational wave obeying the ordinary wave equation

$$\frac{\partial^2}{\partial t^2}(B^\mu)^\cdot + \Delta^2 (B^\mu)^\cdot = 0.$$

This wave, being transversely polarized, has only one nonzero component in the plane perpendicular to the direction of propagation.

With the aid of the gravitational-wave solution, one can lower the order of the subsystem for "vortical" additions and obtain an equation for a general-relativistic Alfvén perturbation. However, the latter exhibits wave properties only for sufficiently small sizes of the perturbed region

$$\sqrt{\frac{2W}{p + \varepsilon + 2W}}\,\frac{k}{b} \gg 1.$$

Whereas small-scale Alfvén waves develop much more rapidly than the changes with time of the unperturbed model (1), Alfvén perturbations with scale

$$\frac{2W}{\varepsilon + p + 2W} \frac{k^2}{b^2} \lesssim 1$$

have a characteristic evolution time of the order of the characteristic time of variation of the scale factors $a(t)$ and $b(t)$. It makes no sense to speak separately of gravitational and Alfvén waves for wavelengths greater than the "horizon" $1/\sqrt{\chi(\varepsilon + W)}$, since for such scales the "vortical" subsystem does not split into Alfvén and gravitational waves.

The system of linear equations for the "potential" terms in the perturbations δu^μ, δH^μ, h^3_μ, h^ν_μ, and also for the perturbations $\delta\varepsilon$, δu^3, δH^3, h^3_3 (we shall call it system II) admits a certain exact solution corresponding to the arbitrariness in the choice of the synchronous reference frame in the perturbed Riemannian space. With the help of this "fictitious" solution for the perturbations, one can reduce the order of system II to 6 in the presence of a magnetic field and to 5 in its absence.*

If, for system II, one looks for solutions with inhomogeneity scales much smaller than the "horizon" $1/\sqrt{\chi(\varepsilon + W)}$, then in the first approximation in $1/\Delta t$ one can single out from it the solution corresponding to a gravitational wave of "potential" type. In a synchronous reference frame such a gravitational wave will be longitudinal, since the tensor $(h^\alpha_\beta)^\bullet$ has a nonzero component only along the direction of propagation of the wave. Using the solution for the longitudinal gravitational wave, one can reduce the order of system II by 2. After this we obtain, with accuracy up to terms of order $1/\Delta t$ (in the coefficients), the equations of relativistic magnetohydrodynamics with gravitation (RMHD) in the cosmological model (1). Analysis of the resulting system of RMHD equations leads to the following conclusions. In the presence of a cosmological magnetic field, the minimal sizes of matter inhomogeneities held together by gravitation will be different from those in isotropic models, where one may use the Jeans criterion. If along the field the critical size of an inhomogeneity is of the Jeans order

$$\sqrt{\frac{dp}{d\varepsilon} \Big/ \chi(\varepsilon + W)},$$

then across the magnetic field the critical size of a condensation of matter is of the order

$$\sqrt{\frac{dp}{d\varepsilon} + \frac{2W}{p + \varepsilon + 2W}\left(1 - \frac{dp}{d\varepsilon}\right) \Big/ \chi(\varepsilon + W)}.$$

For sizes of matter inhomogeneities smaller than these critical sizes, the perturbations of the matter and magnetic field split into linear magnetohydrodynamic fast and slow waves over times much shorter than the characteristic time of variation of the basic model (1); for the phase velocity v_φ of magnetohydrodynamic waves one obtains the relativistic dispersion equation

$$v_\varphi^4 - v_\varphi^2 \left[\frac{(p+\varepsilon)\dfrac{dp}{d\varepsilon} + 2W + \dfrac{dp}{d\varepsilon} 2W \cos^2\theta}{p+\varepsilon+2W} \right] + \frac{2W \dfrac{dp}{d\varepsilon} \cos^2\theta}{p+\varepsilon+2W} = 0,$$

where θ is the angle between the direction of propagation of the wave and the magnetic field. The development of a condensation of matter of sizes greater than these critical sizes occurs over times of the order of the development time of the unperturbed model. The character of the increase of the density of matter in condensations of sizes greater than the critical ones depends essentially on the configuration of the initial perturbation (i.e., on the ratio w/k), if the size of the latter is smaller than the "horizon."

* In the absence of a magnetic field, vortices in particles are conserved:

$$a^2 b(p+\varepsilon)\left[\nabla_\alpha \delta u_\beta - \nabla_\beta \delta u_\alpha\right] = \text{const},$$

where $\alpha, \beta = 1, 2, 3$. System II therefore has an integral corresponding to the conservation of the vortex components perpendicular to the axis of symmetry.

** For perturbation sizes $\gg$ the "horizon," gravitational waves as an independent object do not exist in anisotropic models; for such scales, in the general theory of relativity, in order to solve the problem of density perturbations one must specify two arbitrary functions more than in Newtonian theory (see the last part of this article).

In conclusion we shall present the results concerning the characterization of the development of density inhomogeneity $\delta\varepsilon/\varepsilon$ on a scale larger than the "horizon" for dust and an ultrarelativistic gas near singular instants of time. In this case all unknown quantities in system II can be sought in the form of series in powers of the wave numbers k, w, with coefficients depending on time. The presence of a magnetic field does not change, in the first approximation in $\chi\varepsilon t^2 \ll 1$, the laws of density growth; therefore below the 6th arbitrary function in the solution of the Cauchy problem near collapse into a "pancake" for $\delta\varepsilon/\varepsilon$ is omitted. We write the form of the density contrast $\delta\varepsilon/\varepsilon$ for $\chi\varepsilon t^2 \ll 1$ near singularities of both types. For a "string" $(a = a_0 t^{2/3},\ b = b_0 t^{-1/3})$

$$p = 0: \quad \frac{\delta\varepsilon}{\varepsilon} = \frac{w^2}{a_0^2}\left(C_1 \ln|t| + C_2 t^{-1/3}\right) + C_3 + \frac{w^2}{a_0^2} C_4 t^{2/3} + \frac{k^2}{b_0^2} C_5 t^{5/3},$$

$$p=\frac{\varepsilon}{3}:\quad \frac{\delta\varepsilon}{\varepsilon}=\frac{w^2}{a_0^2}C_1\ln|t|+C_2+\frac{w^2}{a_0^2}\left(C_3t^{2/3}+C_4t^{4/3}\right)+\frac{k^2}{b_0^2}C_5t^2.$$

Using the asymptotics $a=a_0(1+\frac{1}{2}\chi\varepsilon_0 t)$, $b=b_0t$, $\varepsilon=\varepsilon_0/t$ for dust and $a=a_0(1+\frac{3}{4}\chi\varepsilon_0 t^{2/3})$, $b=b_0t$, $\varepsilon=\varepsilon_0 t^{-4/3}$ for an ultrarelativistic gas near a singularity of the "pancake" type, we have

$$p=0:\quad \frac{\delta\varepsilon}{\varepsilon}=\frac{w^2}{a_0^2}\frac{k^2}{b_0^2}C_1\frac{\ln|t|}{t}+\frac{C_2}{t}+C_3+\frac{w^2}{a_0^2}\left(C_4t+C_5t^2\right);$$

$$p=\varepsilon/3:\quad \frac{\delta\varepsilon}{\varepsilon}=\frac{w^2}{a_0^2}\frac{\ln|t|}{t^{2/3}}C_1+\frac{C_2}{t^{2/3}}+C_3+\frac{w^2}{a_0^2}C_4t^{4/3}+C_5t^{2/3}.$$

It follows from this that singular collapses into a "string" and a "pancake" are always unstable, since as the singular instant of time is approached $\delta\varepsilon/\varepsilon\to\infty$. Although the expression for $\delta\varepsilon/\varepsilon$ also contains perturbations that remain bounded while passing through the singular instant, they cannot be used to explain the formation of primary gravitationally isolated objects, because singular configurations of the "string" and "pancake" type are unrealizable (unstable). Let us recall that in Friedmann models density perturbations in an ultrarelativistic gas are bounded when passing through the singularity, and for large scales:

$$\delta\varepsilon/\varepsilon=C_1\sqrt{|t|}+C_2tn^2.$$

These results correct the conclusions of article ([3]), in which the laws of development of large-scale density inhomogeneities near singularities for dust and gases $p=\varepsilon/3$ and $p=\varepsilon$ were written out taking into account only one initial perturbation out of five. A detailed procedure for obtaining the results presented here can be found in the dissertation ([4]).

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