THE GAMMA-CORRELATION PROCESS AND ITS PROPERTIES

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Abstract

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MATHEMATICS

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THE GAMMA-CORRELATION PROCESS AND ITS PROPERTIES

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1. Theorem. The function $p(t_1, x; t_2, y)$, defined for $0 < t_1 < t_2$, $x, y \geq 0$ by the bilinear expansion

$$p(t_1, x; t_2, y) = p(t_1, x)p(t_2, y) \sum_{k=0}^{\infty} a_k(t_1, t_2) L_k^{\alpha(t_1)} \left(\frac{x}{\beta(t_1)} \right) L_k^{\alpha(t_2)} \left(\frac{y}{\beta(t_2)} \right), \quad (1)$$

where

$$p(t, x) = x^{\alpha(t)} e^{-x/\beta(t)} \Big/ [\beta(t)]^{\alpha(t)+1} \Gamma(\alpha(t) + 1); \quad (2)$$

$$a_k(t_1, t_2) =$$

$$= [\omega(t_1)/\omega(t_2)]^k \left[\frac{\Gamma(\alpha(t_2) + 1)\Gamma(\alpha(t_1) + k + 1)}{\Gamma(\alpha(t_1) + 1)\Gamma(\alpha(t_2) + k + 1)} \right]^{1/2}; \quad (3)$$

$\beta(t) > 0$; $\alpha(t) > -1$ is a monotonically increasing continuous function, $\omega(t)$ is a continuous strictly increasing function, and $L_k^{\alpha}(x/\beta)$ are Laguerre polynomials, is a two-dimensional density defining a Markov process $\xi(t) \geq 0$ for $t > 0$.

Proof. By virtue of the orthogonality and normality of the Laguerre polynomials

$$L_k^{\alpha(t)} \left(\frac{x}{\beta(t)} \right) = \left[\frac{\Gamma(\alpha(t) + 1)\Gamma(\alpha(t) + k + 1)}{k!} \right]^{1/2} \sum_{r=0}^{k} \frac{(-1)^r \binom{k}{r}}{\Gamma(\alpha(t) + r + 1)} \left(\frac{x}{\beta(t)} \right)^r \quad (4)$$

with weight (2) on the half-axis $0 \leq x < \infty$, the function (1) satisfies the following Markov equation for the two-dimensional density:

$$p(t_1, x; t_2, y) = \int_0^\infty \frac{p(t_1, x; t, z)\, p(t, z; t_2, y)}{p(t, z)}\, dz, \tag{5}$$

where

$$p(t, z) = \int_0^\infty p(t_1, x; t, z)\, dx$$

is the marginal density (2) of the gamma distribution, $0 < t_1 < t < t_2$.

The series (1) for $t_1 < t_2$ converges in the mean, since the squares of the coefficients of the series

$$a_k^2(t_1, t_2) = [\omega(t_1)/\omega(t_2)]^{2k}$$

are less than the terms of a geometric progression with ratio $\omega^2(t_1)/\omega^2(t_2) < 1$.

Consequently, to prove the validity of the assertion of the theorem, it remains to show the nonnegativity of the sum of the series (1) for $t_1 < t_2$ and $x, y \geq 0$.

Noting that the Fourier transform of polynomial (4) with weight (2) has the form[1]

$$[\Gamma(\alpha + k + 1)/\Gamma(\alpha + 1)k!]^{1/2} \frac{(-i\tau\beta)^k}{(1 - i\tau\beta)^{\alpha+k+1}},$$

we find the two-dimensional Fourier transform of function (1)

$$\varphi(\tau_1, \tau_2) = \frac{[1 - i\tau_2\beta(t_2)]^{\alpha(t_1)-\alpha(t_2)}}{[1 - i\tau_1\beta(t_1) - i\tau_2\beta(t_2) - (1 - \omega(t_1)/\omega(t_2))\, \tau_1\tau_2\beta(t_1)\beta(t_2)]^{\alpha(t_1)+1}}. \tag{6}$$

Expression (6) is a two-dimensional characteristic function, since the numerator is a one-dimensional characteristic function of the gamma distribution with nonnegative parameter $\alpha(t_2) - \alpha(t_1)$, and the second factor is the two-dimensional characteristic function of the symmetrized gamma correlation studied in [2]. Consequently, (1) is a two-dimensional density. The theorem is proved.

2. In [3] it is noted that the correlation function of a continuous Markov process for $t_1 < t_2$ must have the form $R(t_1, t_2) = \psi(t_1)/\psi(t_2)$, where $\psi(t)$ is a continuous strictly increasing function.

[1] The visible page contains the marker "(1)" after "has the form"; it is preserved here as a footnote marker.

In our case $\psi(t) = \omega(t)[\alpha(t) + 1]^{1/2}$, and

$$R(t_1, t_2) = a_1(t_1, t_2) = \omega(t_1)[\alpha(t_1) + 1]^{1/2}/\omega(t_2)[\alpha(t_2) + 1]^{1/2}, \tag{7}$$

since, in general, $a_k(t_1, t_2)$ is the correlation coefficient between $L_k^{\alpha(t_1)}(\xi(t_1)/\beta(t_1))$ and $L_k^{\alpha(t_2)}(\xi(t_2)/\beta(t_2))$; hence $a_1(t_1, t_2)$ is the correlation coefficient between $\xi(t_1)$ and $\xi(t_2)$.

Definition. The Markov process $\xi(t)$ specified by density (1) will be called a **gamma-correlation process** or a **gamma process**.

3. If $\alpha(t) = \alpha$, $\beta(t) = \beta$ do not depend on t, and $\omega(t) = e^{\lambda t}$, where λ is a positive constant, then $a_k(t_1, t_2) = \exp\{-\lambda(t_2 - t_1)k\}$, and (1) defines the stationary Markov process studied in [4] (for $\beta = 1$).

If $\alpha = -1/2$ (does not depend on t), $\beta(t) = \omega(t) = t$, then $a_k(t_1, t_2) = (t_1/t_2)^k$, and we obtain a gamma process $\xi(t)$ with density

$$\frac{(x/t_1)^{-1/2}e^{-x/t_1}}{\Gamma(1/2)t_1}\frac{(y/t_2)^{-1/2}e^{-y/t_2}}{\Gamma(1/2)t_2}\sum_{k=0}^{\infty}\left(\frac{t_1}{t_2}\right)^k L_k^{-1/2}\left(\frac{x}{t_1}\right)L_k^{-1/2}\left(\frac{y}{t_2}\right) =$$

$$= \frac{\operatorname{ch}\left[2(xy)^{1/2}/(t_2 - t_1)\right]}{\pi\left[t_1(t_2 - t_1)xy\right]^{1/2}}\exp\left\{-\frac{x/t_1 + y/t_2}{1 - t_1/t_2}\right\}. \tag{8}$$

It is easy to verify that, if one considers the Wiener process $\eta(t)$ with parameters

$$\mathbf{M}\eta(t) = 0, \qquad \mathbf{D}\eta(t) = t, \qquad R(t_1, t_2) = (t_1/t_2)^{1/2},$$

then (8) is the two-dimensional density of the process $\xi(t) = \eta^2(t)/2$.

Definition. If, for the process $\xi(t)$ with density (1), the correlation function $R(t_1, t_2) = t_1/t_2$ is equal to the square of the correlation function of the Wiener process, then the process $\xi(t)$ will be called a **gamma process of Wiener type**.

4. Let us consider a more general case of a gamma process of Wiener type. Put $\alpha(t) + 1 = t$, $\omega(t) = t^{1/2}$, $\beta(t) = 1$; then the process will have the basic parameters

$$\mathbf{M}\xi(t) = t, \qquad \mathbf{D}\xi(t) = t, \qquad R(t_1, t_2) = t_1/t_2. \tag{9}$$

The marginal density has the form

$$p(t, x) = x^{t-1}e^{-x}/\Gamma(t), \qquad t > 0, \qquad x \geq 0.$$

The two-dimensional density is written in the form of a series

$$p(t_1, x; t_2, y) = \frac{x^{t_1-1}e^{-x}}{\Gamma(t_1)} \frac{y^{t_2-1}e^{-y}}{\Gamma(t_2)} \sum_{k=0}^{\infty} \left(\frac{t_1}{t_2}\right)^{k/2} \left[\frac{\Gamma(t_2)\Gamma(t_1+k)}{\Gamma(t_1)\Gamma(t_2+k)}\right]^{1/2} L_k^{t_1-1}(x) L_k^{t_2-1}(y), \tag{10}$$

where $0 < t_1 < t_2$.

Let us note that the trajectories of the process defined by density (10) may, as in the Wiener case, be regarded as issuing from the origin; they all lie in the first quadrant of the plane XOT, where time t is plotted along the abscissa axis. The mean value $M\xi(t) = t$ grows without bound as t increases, whereas in the Wiener case $M\eta(t) = 0$.

In conclusion, we write down some other characteristics of the process (10). The characteristic function of the two-dimensional distribution has the form

$$\varphi(\tau_1, \tau_2) = \frac{1}{(1 - i\tau_2)^{t_2 - t_1}} \frac{1}{\left[1 - i\tau_1 - i\tau_2 - (1 - (t_1/t_2)^{1/2})\, \tau_1\tau_2\right]^{t_1}}, \quad 0 < t_1 < t_2.$$

The initial moments of the marginal distribution are expressed by the formula

$$M\xi^k(t) = t(t+1)\cdots(t+k-1), \qquad k = 1, 2, \ldots$$

The conditional moments $m_k(t_1, t_2, x)$ of the quantity $\xi(t_2)$ under the condition $\xi(t_1) = x$ are expressed as follows:

$$m_k(t_1, t_2, x) = \Gamma(t_2 + k) \sum_{\mu=0}^{k} \frac{\binom{k}{\mu} \left[x(t_1/t_2)^{1/2}\right]^{\mu}}{\Gamma(t_2 + \mu)} \sum_{\nu=0}^{k-\mu} \binom{k-\mu}{\nu} \left[(t_1/t_2)^{1/2}\right]^{\nu} \times$$

$$\times \frac{\Gamma(t_2+\mu)\Gamma(t_1+\mu+\nu)}{\Gamma(t_1+\mu)\Gamma(t_2+\mu+\nu)}. \tag{11}$$

With the aid of (11) one can find the conditional variance

$$D(\xi(t_2) \mid \xi(t_1) = x) = t_2 - t_1 + t_1 \left[1 - (t_1/t_2)^{1/2}\right]^2 + 2(t_1/t_2)^{1/2} \left[1 - (t_1/t_2)^{1/2}\right] x. \tag{12}$$

Remark. We have put $\beta = 1$ only to shorten the notation; replacing x and y respectively by $x/\beta(t_1)$ and $y/\beta(t_2)$ will not change the correlation function and will make it possible, without difficulty, to pass to a more general process of Wiener type.

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Note: Figure translations are in progress. See original paper for figures.