ANALOGUES OF THE CAUCHY KERNEL AND THE RIEMANN BOUNDARY-VALUE PROBLEM ON A HYPERELLIPTIC SURFACE

Mathematics

1970

SovietRxiv

Abstract

Full Text

UDC 517.948.32:517.544

Mathematics

E. I. Zverovich

ANALOGUES OF THE CAUCHY KERNEL AND THE RIEMANN BOUNDARY-VALUE PROBLEM ON A HYPERELLIPTIC SURFACE

(Presented by Academician P. Ya. Kochina on May 5, 1969)

In ($^{1-3}$) the existence of analogues of the Cauchy kernel was proved, and in ($^{3-9}$) the index was computed, the number of solutions and the solvability conditions for the Riemann boundary-value problem on closed Riemann surfaces were found. The solutions themselves of the Riemann problem and analogues of the Cauchy kernel have so far been constructed explicitly only in the case of a torus (^{10}), given by an algebraic equation.

In the present note, analogues of the Cauchy kernel are constructed explicitly on the Riemann surface $\mathfrak{R}$ of genus 2, given by the algebraic equation:

$$w^2 = f(z), \quad \text{where } f(z) = (1-z^2)(1-k^2z^2)(1-h^2z^2), \quad 0<h<k<1. \tag{1}$$

On this basis, a solution in closed form is given for the Riemann boundary-value problem on $\mathfrak{R}$.

1. We shall write points of the surface $\mathfrak{R}$ in the form of ordered pairs (z, w), (τ, ζ), (t, v), (ρ, ξ), ... of complex numbers connected by equation (1). By $+\sqrt{f(z)}$ we denote the continuous branch of the root, single-valued in the z-plane cut along the rectilinear segments $[-1/h, -1/k]$, $[-1, 1]$, $[1/k, 1/h]$, defined by the condition: $\lim_{z\to 0+0\cdot i}(+\sqrt{f(z)}) = +1$. The sets of points of the surface $\mathfrak{R}$ of the form $(z, +\sqrt{f(z)})$ and $(z, -\sqrt{f(z)})$ will be called, respectively, its upper and lower sheets. The sheets are glued along the indicated segments. On each sheet there is a point at infinity. We denote these points respectively by $(\infty, +\infty)$, $(\infty, -\infty)$. The canonical cuts A_1, A_2, B_1, B_2 of the surface $\mathfrak{R}$ are chosen and oriented as indicated in Fig. 1. The dashed lines here are those lying on the lower sheet.

Fig. 1

In the computations we agree to contract the canonical cuts to doubly traversed rectilinear segments joining the branch points. A basis of Abelian differentials of the first kind (^{11,12}) on $\mathfrak{R}$ is formed by the differentials

$$d\tau/\zeta, \qquad \tau\, d\tau/\zeta, \qquad (\tau,\zeta)\in\mathfrak{R}. \tag{2}$$

Introducing the notation

$$\int_{1/k}^{1/h}\frac{d\tau}{|\zeta|}=K, \qquad \int_{1/k}^{1/h}\frac{\tau\, d\tau}{|\zeta|}=H, \qquad \int_{1}^{1/k}\frac{d\tau}{|\zeta|}=K', \qquad \int_{1}^{1/k}\frac{\tau\, d\tau}{\zeta}=H', \tag{3}$$

it is easy to compute the period matrix of the differentials (2). We shall need a complex-normalized basis of differentials of the first kind on $\mathfrak{R}$, i.e. such a

basis whose A-periods form the identity matrix. This basis is expressed through (2) and (3) by the formulas:

$$du_1(\tau,\zeta)=-\frac{1}{4K}\frac{d\tau}{\zeta}-\frac{1}{4H}\frac{\tau\, d\tau}{\zeta}; \qquad du_2(\tau,\zeta)=\frac{1}{4K}\frac{d\tau}{\zeta}-\frac{1}{4H}\frac{\tau\, d\tau}{\zeta}. \tag{4}$$

The period matrix of this basis has the form

$$\left\|\begin{matrix} 1 & 0 & \frac{i}{2}\left(\frac{H'}{H}+\frac{K'}{K}\right) & \frac{i}{2}\left(\frac{H'}{H}-\frac{K'}{K}\right) \\ 0 & 1 & \frac{i}{2}\left(\frac{H'}{H}-\frac{K'}{K}\right) & \frac{i}{2}\left(\frac{H'}{H}+\frac{K'}{K}\right) \end{matrix}\right\|. \tag{5}$$

2. As the simplest analogue of the Cauchy kernel on $\mathfrak{R}$ one may take the expression

$$\frac{w+\zeta}{2\zeta}\frac{d\tau}{\tau-z}, \qquad (z,w)\in\mathfrak{R}, \quad (\tau,\zeta)\in\mathfrak{R}. \tag{6}$$

Let us formulate its properties.

1) With respect to the variable (τ,ζ), expression (6) is an Abelian differential of the third kind on $\mathfrak{R}$ with three simple poles at the points (z,w), $(\infty,\pm\infty)$. The residue at the first point is $(+1)$, and at the other two it is $(-1/2)$.

2) With respect to the variable (z,w), expression (6) is a meromorphic function everywhere on $\mathfrak{R}$ with a simple pole at the point (τ,ζ). In addition, there are two poles of order 2 at the points $(\infty,\pm\infty)$, near which the expansion

$$\frac{w+\zeta}{2\zeta}\,\frac{d\tau}{\tau-z}=\pm\frac{ikh}{2}\left(\frac{d\tau}{\zeta}z^2+\frac{\tau\,d\tau}{\zeta}z\right)+\ldots, \tag{7}$$

takes place, where the dots denote regular terms. From (7) it is clear that the coefficients of the positive powers of z form the basis (2) of differentials of the first kind.

Properties 1) and 2) are easily proved by expanding expression (6) in powers of the local parameter in neighborhoods of the indicated poles. Here, in neighborhoods of the points $(\infty,\pm\infty)$, the variable $\sigma=1/\tau$ is used as a local parameter of the point (τ,ζ). In neighborhoods of branch points, for example $(\tau,\zeta)=(0,1)$, the variable $\sigma=\sqrt{\tau-1}$ is used as a local parameter. In neighborhoods of the remaining points (τ,ζ), the local parameter is taken to be τ. The same remarks apply to the variable (z,w).

Let L be a finite smooth curve on $\mathfrak{R}$, and let $g(\tau,\zeta)$ be an H-continuous function of the points of this curve. The Cauchy-type integral with kernel (6)

$$\Phi(z,w)=\frac{1}{2\pi i}\int_L\frac{w+\zeta}{2\zeta}\,\frac{g(\tau,\zeta)}{\tau-z}\,d\tau,\qquad (z,w)\in\mathfrak{R}-L, \tag{8}$$

represents a function $\Phi(z,w)$, analytic on $\mathfrak{R}-L$, admitting at the points $(z,w)=(\infty,\pm\infty)$ poles of order 2. From (7) the principal parts of the expansions of the function $\Phi(z,w)$ in neighborhoods of the points $(\infty,\pm\infty)$ are easily found. These expansions show that the equalities

$$\int_L g(\tau,\zeta)\frac{d\tau}{\zeta}=0,\qquad \int_L g(\tau,\zeta)\frac{\tau\,d\tau}{\zeta}=0 \tag{9}$$

ensure analyticity of the function $\Phi(z,w)$ everywhere on $\mathfrak{R}-L$. From property 1) there follows the relation

$$\frac{w+\zeta}{2\zeta}\,\frac{d\tau}{\tau-z}=\frac{d\tau}{\tau-z}+\ldots\quad\text{as }(\tau,\zeta)\to(z,w). \tag{10}$$

It follows from this relation that $\Phi(z,w)$ is H-continuously extendable to L, and its limiting values $\Phi^+(t,v)$ (from the left) and $\Phi^-(t,v)$ (from the right) on L are related by the Sokhotski formulas:

$$\Phi^\pm(t,v)=\pm\frac{1}{2}g(t,v)+\frac{1}{2\pi i}\int_L\frac{v+\zeta}{2\zeta}\,\frac{g(\tau,\zeta)}{\tau-t}\,d\tau,\qquad (t,v)\in L, \tag{11}$$

where the integral is understood in the sense of the principal value.

3. For constructing the solution of the Riemann boundary-value problem on $\mathfrak{R}$, a single kernel (6) is not sufficient. We shall give several other analogues of the Cauchy kernel.

The "discontinuous" analogue of the Cauchy kernel:

$$d\omega_{(z,w)}(\tau,\zeta)=\frac{1}{8KH}\begin{vmatrix}\dfrac{\omega+\zeta}{2\zeta}\,\dfrac{d\tau}{\tau-z} & \displaystyle\int_{A_1}\frac{w+\xi}{2\xi}\,\frac{d\rho}{\rho-z} & \displaystyle\int_{A_2}\frac{w+\xi}{2\xi}\,\frac{d\rho}{\rho-z}\\ \dfrac{d\tau}{\zeta} & -2K & 2K\\ \dfrac{\tau\,d\tau}{\zeta} & -2H & -2H\end{vmatrix}. \qquad (12)$$

The algebraic complement of the element

$$\frac{w+\zeta}{2\zeta}\,\frac{d\tau}{\tau-z}$$

of the determinant is equal to $8KH$; therefore, with respect to the variable (τ,ζ), expression (12) has exactly the same properties as (6). In particular, relation (10) remains valid. With respect to the variable (z,w), the integrals in the first row of determinant (12) have discontinuities of the first kind along the curves A_1 and A_2, respectively. This follows from the fact that the integrals may be obtained by formula (8), where one should take $g(\tau,\zeta)=2\pi i$, and as L use, respectively, the curves A_1 and A_2. If the first row of determinant (12) is expanded in powers of z in a neighborhood of the points $(\infty,\pm\infty)$ and (7) is used, then the corresponding values are found to be $\pm 2\pi i\,du_1(\tau,\zeta)$ and $\pm 2\pi i\,du_2(\tau,\zeta)$, where the latter (z,w) expression (12) has a unique simple pole at the point $(z,w)=(\tau,\zeta)$ and discontinuities of the first kind along the curves A_1 and A_2. Using formulas (11), it is easy to compute the increments that the kernel (12) receives when the point (z,w) passes through the lines A_1 and A_2. These increments turn out to be, respectively, equal to $\pm 2\pi i\,du_1(\tau,\zeta)$ and $\pm 2\pi i\,du_2(\tau,\zeta)$, where the latter differentials are computed by formulas (4). For the Cauchy-type integral with kernel (12), Sokhotski formulas analogous to formulas (11) are valid. The absence in the kernel (12), for $(z,w)\neq(\tau,\zeta)$, of poles makes it convenient for constructing the solution of the homogeneous Riemann problem.

For constructing the solution of the nonhomogeneous Riemann problem, "meromorphic" analogues of the Cauchy kernel are needed, i.e. such analogues which, with respect to the variable (z,w), have no lines of discontinuity, but have poles at a finite number of prescribed points. The existence of such kernels is proved in (1). For the Riemann surface $\mathfrak{R}$ given by equation (1), every meromorphic analogue of the Cauchy kernel is expressed explicitly through (6) and (2). Having no space for proving this assertion, we restrict ourselves to an example. On

$\mathfrak{R}$ take two points (α, a) and (β, b), finite, distinct from the branch points, and such that $\alpha \neq \beta$. The expression

$$\frac{d\tau}{2(\alpha-\beta)\zeta} \begin{vmatrix} \dfrac{w+\zeta}{\tau - z} & \dfrac{w+a}{\alpha - z} & \dfrac{w+b}{\beta - z} \\ \tau & \alpha & \beta \\ 1 & 1 & 1 \end{vmatrix} \tag{13}$$

is a meromorphic analogue of the Cauchy kernel, having, with respect to the variable (z, w), simple poles at the points (τ, ζ), (α, a), (β, b), and regular at the remaining points of the surface $\mathfrak{R}$. With respect to the variable (τ, ζ), expression (13) has exactly the same properties as (6). In particular, for the Cauchy-type integral with kernel (13), the Sokhotski formulas are valid, and the integral itself is a piecewise-meromorphic function on $\mathfrak{R}$, having two simple poles at the points (α, a) and (β, b).

4. Consider the homogeneous Riemann boundary-value problem on $\mathfrak{R}$. We seek a function $\Phi(z, w)$, analytic on $\mathfrak{R} - L$, H-continuously extendable to L, under the boundary condition:

$$\Phi^+(t, v) = G(t, v)\Phi^-(t, v), \qquad (t, v) \in L, \tag{14}$$

where L is a closed, smooth, finite, oriented curve, and $G(t, v) \neq 0$ is a given H-continuous function of points of this curve. The mutual position of the curve L and the canonical cuts on $\mathfrak{R}$ is immaterial. Fix on L an arbitrary point (t_0, v_0). Select a branch of the function $\ln G(t, v)$ with a discontinuity at the point (t_0, v_0), and assume that this branch enters into all the formulas given below. Introduce the integer $\varkappa$ equal to

$$\varkappa = \operatorname{Re}\left\{\frac{1}{2\pi i}\left[\ln G(t_0 - 0, v_0 - 0) - \ln G(t_0 + 0, v_0 + 0)\right]\right\}.$$

Fix on $\mathfrak{R}$ an arbitrary finite point (z_0, w_0) and denote

$$u_1(z, w) = \int_{(z_0, w_0)}^{(z, w)} du_1(\tau, \zeta), \qquad u_2(z, w) = \int_{(z_0, w_0)}^{(z, w)} du_2(\tau, \zeta), \tag{15}$$

the normalized basis of Abelian integrals of the first kind on $\mathfrak{R}$. We shall assume that in the integrals (15) the path of integration does not cross the canonical cuts. The general solution of problem (14) is given by the formula:

$$\Phi(z, w) = \varphi(z, w) \exp\left\{\frac{1}{2\pi i}\int_L \ln G(\tau, \zeta)\, d\omega_{(z,w)}(\tau, \zeta)\right.$$

$$- \int_{(z_0,w_0)}^{(z_1,w_1)} d\omega_{(z,w)}(\tau,\zeta) - \int_{(z_0,w_0)}^{(z_2,w_2)} d\omega_{(z,w)}(\tau,\zeta) + 2\pi i \left[m_1 u_1(z,w) + m_2 u_2(z,w)\right] \Bigg\} , \tag{16}$$

where $d\omega_{(z,w)}(\tau,\zeta)$ is the discontinuous analogue of the Cauchy kernel (12), and the paths of integration in the integrals from (z_0, w_0) to (z_j, w_j) do not cross the canonical cuts. The points (z_1, w_1), $(z_2, w_2) \in \mathfrak{R}$ and the integers m_1, m_2 are found from the following Jacobi inversion problem:

$$\begin{gathered} u_1(z_1,w_1) + u_1(z_2,w_2) = \frac{1}{2\pi i} \int_L \ln G(\tau,\zeta)\, du_1(\tau,\zeta) + n_1 + \\ + \frac{i}{2}\left(\frac{H'}{H} + \frac{K'}{K}\right) m_1 + \frac{i}{2}\left(\frac{H'}{H} - \frac{K'}{K}\right) m_2; \qquad (17) \\ u_2(z_1,w_1) + u_2(z_2,w_2) = \frac{1}{2\pi i} \int_L \ln G(\tau,\zeta)\, du_2(\tau,\zeta) + n_2 + \\ + \frac{i}{2}\left(\frac{H'}{H} - \frac{K'}{K}\right) m_1 + \frac{i}{2}\left(\frac{H'}{H} + \frac{K'}{K}\right) m_2, \end{gathered}$$

where n_1 and n_2 are also unknown integers which, however, do not enter into formula (16). The function $\varphi(z, w)$ entering into (16) is an arbitrary meromorphic everywhere on $\mathfrak{R}$ (algebraic) function whose divisor is a multiple$^{(3)}$ of the divisor $(t_0, v_0)^{-\varkappa}(z_0, w_0)^{-2}(z_1, w_1)(z_2, w_2)$. It may be taken in the form

$$\varphi(z,w) = [P(z)w + Q(z)]/R(z),$$

where $P(z), Q(z), R(z)$ are polynomials in z, the degrees and coefficients of which are found from the conditions expressing the multiplicity of the function $\varphi(z, w)$ with respect to the divisor $(t_0, v_0)^{-\varkappa}(z_0, w_0)^{-2}(z_1, w_1)(z_2, w_2)$.

5. Using the apparatus of Riemann theta-functions, for solving the inversion problem (17) one can give a simple algorithm reducing the determination of the points (z_1, w_1) and (z_2, w_2) to quadratures and arithmetic operations.

Received
30 IV 1969

CITED LITERATURE

1. S. Ya. Gusman, Yu. L. Rodin, *Sibirsk. Mat. Zh.*, **3**, No. 4 (1962).

2. H. Behnke, K. Stein, *Math. Ann.*, **120**, issue 4 (1949).
3. W. Koppelman, *J. Math. and Mech.*, **10**, No. 2 (1961).
4. R. N. Abdundullaev, DAN, **172**, No. 4 (1967).
5. R. N. Abdullaev, DAN, **160**, No. 5 (1965).
6. Yu. L. Rodin, DAN, **129**, No. 6 (1959).
7. Yu. L. Rodin, DAN, **132**, No. 5 (1960).
8. E. I. Zverovich, G. S. Litvinchuk, UMN, **23**, issue 3 (41) (1968).
9. E. I. Zverovich, *Sibirsk. Mat. Zh.*, **7**, No. 4 (1966).
10. E. I. Zverovich, DAN, **188**, No. 1 (1969).
11. J. Springer, *Introduction to the Theory of Riemann Surfaces*, IL, 1960.
12. N. G. Chebotarev, *Theory of Algebraic Functions*, M.–L., 1948.

Note: Figure translations are in progress. See original paper for figures.