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Abstract

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ON THE THEORY OF THE SECOND VARIATION OF FUNCTIONALS ON THE CLASS S OF UNIVALENT FUNCTIONS

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By S we shall, as usual, denote the class of functions $f(z)$ univalent in the disk $|z| < 1$ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$. We shall write the Taylor expansion of $f(z) \in S$ in the form

$$f(z) = \sum_{0}^{\infty} a_k z^{k+1}, \qquad a_0 = 1.$$

The class S becomes bicompact if one introduces in it the metric

$$\rho(f, g) = \max_{|z|=1/2} |f(z) - g(z)|, \qquad f, g \in S.$$

For the construction of the theory of extremal problems on the class S, it is important to be able, for a given $f(z) \in S$, to find a function $g(z) \in S$ lying in a prescribed ε-neighborhood of f. This can be achieved, for example, by the following known device. A function $h(z)$, regular in some annulus $K_{r_0} = \{z : r_0 < |z| < 1\}$, will be called admissible with respect to $f(z)$ if $f(z) + h(z)$ is regular and univalent in K_{r_0}. Following G. M. Goluzin ([1]), one can show that, by means of a change of variables, the singularities of the function $f + h$ are removed and a function is obtained which is regular and univalent in the disk $|z| < 1$. Such a construction is always possible as soon as h satisfies a certain smallness condition. It is important to emphasize that the regularizing change of variables is made with a known degree of arbitrariness, and this circumstance is useful in various concrete cases.

As $h(z)$ one can always take a function of the form

$$\varepsilon \int_{\mathcal{C}} \frac{f'(z)}{f(z) - f(t)} \left(\frac{t f'(t)}{f(t)} \right)^2 \nu(dt), \tag{1}$$

where $\nu(dt)$ is a complex-valued measure with support $\mathcal{C}$, lying in some disk $|z| \leq r_1 < 1$, and ε is a real constant. As a result of regularization and subsequent normalization we obtain a function $f(z, \varepsilon) \in S$. For fixed $r_0 < 1$, one can indicate an $\varepsilon_0 > 0$ such that, for $|\varepsilon| < \varepsilon_0$ and $|z| \le r_0$, $f(z, \varepsilon)$ is represented in the form of a convergent series

$$f(z, \varepsilon) = \sum_0^\infty \frac{\varepsilon^k}{k!} f_k(z), \qquad f_0(z) = f(z).$$

The function $f_k(z)$, $k = 1, 2, \ldots$, is regular for $|z| < 1$, and we shall call it the k-th variation of the function $f(z)$. It is convenient to denote the first and second variations by $\delta f(z)$, $\delta^2 f(z)$, or by $\delta f(z; \nu)$, $\delta^2 f(z; \nu)$. For the first variation we have the well-known formula of M. Schiffer ([2]),

$$\delta f(z) = \int_{\mathcal{C}} \left[\frac{f'(z)}{f(z) - f(t)} \left(\frac{Df(t)}{f(t)} \right)^2 + \frac{Df(z)\, t}{t - z} - f(z) \right] \nu(dt) +$$

$$+ Df(z) \int_{\mathcal{C}} \frac{\bar{t} z}{1 - \bar{t} z} \nu(dt) + iC_1 (Df(z) - f(z)), \tag{2}$$

where D is the differential operator $z\, d/dz$, and C_1 is an arbitrary real constant.

We shall not give the formula for $\delta^2 f(z)$, in view of its cumbersomeness. The coefficients of the power series of the function $\delta f(z)$ will be called the first variations of the coefficients a_k, $k = 1, 2, \ldots$, and will be denoted, respectively, by δa_k, $k = 1, 2, \ldots$. Thus,

$$\delta f(z) = \sum_1^\infty \delta a_k z^{k+1}.$$

Assuming $w \ne 0$, consider, for small $|z|$, the expansion

$$\frac{f^2(z)}{f(z) - w} = \sum_0^\infty q_k(w) z^{k+1}. \tag{3}$$

It is clear that $q_k(w)$ is a polynomial in $1/w$ of degree k. We also introduce the functions

$$p_k(z) = \sum_{l=1}^{k} (k - l + 1) a_{k-l} z^{-l}, \quad l = 1, 2, \ldots \tag{4}$$

From (2) we find

$$\delta a_k = \int_E \mathfrak{A}_k(z)\, \nu(dz) + \int_E \mathfrak{B}_k(\bar{z})\, \overline{\nu(dz)} + iC_1 k a_k, \quad k = 1, 2, \ldots, \tag{5}$$

where

$$\mathfrak{A}_k(z) = q_k[f(z)]\,(Df(z)/f(z))^2 + p_k(z) + ka_k,$$

$$\mathfrak{B}_k(z) = p_k(1/z), \quad k = 1, 2, \dots \tag{6}$$

It is not hard to verify that $\mathfrak{A}_k(z), \mathfrak{B}_k(z),\ k = 1, 2, \dots$, are regular functions in the disk $|z| < 1$, and moreover $\mathfrak{A}(0) = \mathfrak{B}(0) = 0$.

Taking admissible functions in the form (1), in a prescribed neighborhood U of the element $f,\ U = \{g : \rho(f, g) \leq \varepsilon_0\}$, we obtain a fairly large set. Therefore, with the help of variations of this kind one can resolve the question of necessary conditions for a local extremum of a functional. If one considers the question of sufficient conditions, then it is necessary to use admissible functions of a more general form than (1).

We shall give the necessary conditions for a local extremum, restricting ourselves to functionals of the form $J(f) = F(a_1, \dots, a_n)$, where F is a twice continuously differentiable function of its arguments, defined in the ball

$$\sum_1^n |a_k|^2 \leq R^2$$

of sufficiently large radius. One of the necessary conditions for an extremum, namely the M. Schiffer equation, is well known. Assuming that the functional $J(f)$ is stationary at the function f, with the help of (5) we obtain, by virtue of the arbitrariness of the measure $\nu(dz)$ and of the constant C_1, the conditions

$$\sum_1^n \left[\bar{\lambda}_k \mathfrak{A}_k(z) + \lambda_k \overline{\mathfrak{B}_k(z)}\right] = 0, \tag{7}$$

$$\operatorname{Im} \sum_1^n \bar{\lambda}_{kka} k = 0, \tag{8}$$

where $\lambda_k = 2\partial F/\partial \bar{a}_k,\ k = 1, \dots, n$. Applying (6), relation (7) can be written in the form

$$Q(f(z))\,(Df(z)/f(z))^2 + P(z) = 0, \tag{9}$$

where

$$Q(w) = \sum_1^n \bar{\lambda}_k q_k(w), \quad P(z) = \sum_1^n \left[\bar{\lambda}_k (p_k(z) + ka_k) + \lambda_k \overline{p_k\left(\frac{1}{z}\right)}\right]. \tag{10}$$

Equation (9) is precisely the M. Schiffer equation. From (8) we obtain that $\operatorname{Im} P(z) = 0$, if $|z| = 1$.

If the functional J has a local maximum at the function f, then the second variation of the functional must be nonpositive. For convenience of the computations, we first consider the case when F is a linear function

$$F = \operatorname{Re} \sum_{1}^{n} \overline{\lambda_k} a_k.$$

Then

$$\delta^2 F = \operatorname{Re} \sum_{1}^{n} \overline{\lambda_k} \delta^2 a_k.$$

Omitting all intermediate calculations, we give the final formula for $\delta^2 F$. We first introduce a number of notations. Put

$$\mathcal{K}(z,t) = [Df(t)/(f(z) - f(t))]^2 - (Df(t)/f(t))^2 - zt/(z-t)^2, \qquad (11)$$

and also

$$P^*(z) = \sum_{1}^{n} (k+1) \left[\overline{\lambda_k} p_k(z) - \lambda_k \overline{p_k\left(\frac{1}{z}\right)} \right], \quad R(z) = \sum_{1}^{n} \overline{\lambda_k} p_k(z). \qquad (12)$$

Next put

$$Q^*(w,z) = \sum_{k=2}^{n} \overline{\lambda_k} \sum_{l=1}^{k-1} (k-l+1) z^{-l} q_{k-l}(w), \quad Q^*(w) = \sum_{1}^{n} k \overline{\lambda_k} q_k(w). \qquad (13)$$

We now introduce two kernels: the symmetric kernel

$$\begin{aligned} \mathfrak{A}(z,t) = {} & P(z)\mathcal{K}(z,t) + P(t)\mathcal{K}(t,z) + [zP^*(t) - tP^*(z)]/(z-t) + \\ & + (Df(z)/f(z))^2 [Q^*(f(z),t) + Q^*(f(z))] + (Df(t)/f(t))^2 [Q^*(f(t),z) + \\ & + Q^*(f(t))] - R(z) - R(t) + \sum_{1}^{n} \overline{\lambda_k} k^2 a_k \end{aligned} \qquad (14)$$

and the Hermitian kernel

$$\mathfrak{B}(z,\bar{t}) = \frac{z\bar{t}}{(1-z\bar{t})^2}[P(z)+\overline{P(t)}] - \frac{z\bar{t}}{1-z\bar{t}}[P^*(z)+\overline{P^*(t)}] - \left(\frac{Df(z)}{f(z)}\right)^2 \times \times Q^*\left[f(z),\frac{1}{\bar{t}}\right] - \overline{\left(\frac{Df(t)}{f(t)}\right)^2 Q^*\left[f(t),\frac{1}{\bar{z}}\right]} + R\left(\frac{1}{\bar{z}}\right) + R\left(\frac{1}{t}\right). \tag{15}$$

Introduce one more function

$$\mathfrak{C}(z) = (Df(z)/f(z))^2 Q^*[f(z)] + P^*(z) - R(z) + \overline{R(1/\bar{z})} + \sum_1^n \overline{\lambda_k} k^2 a_k. \tag{16}$$

Then

$$\delta^2 F = \mathrm{Re}\left\{\iint_{\mathcal{C}\mathcal{C}} \mathfrak{A}(z,t)\nu(dz)\nu(dt) - \iint_{\mathcal{C}\mathcal{C}} \mathfrak{B}(z,\bar{t})\nu(dz)\overline{\nu(dt)} + + 2iC_1 \int_{\mathcal{C}} \mathfrak{C}(z)\nu(dz) - C_1^2 \sum_1^n \overline{\lambda_k} k^2 a_k \right\}. \tag{17}$$

In order to obtain the second variation in the general case, one must add to expression (17) a quadratic form in δa_k, $k = 1, 2, ...$, whose coefficients are computed from the second derivatives of the function F.

Using the arbitrariness in the choice of the measure $\nu(dz)$ and of the constant C_1, it is easy to obtain from the condition $\delta^2 F \leqslant 0$ that

$$P(z) \geqslant 0 \quad \text{for } |z| = 1. \tag{18}$$

Definition 1. A univalent function $f(z) \in S$ satisfying equation (9), in which for $P(z)$ condition (19) is fulfilled, will be called an **extremal univalent function**.

By S_n we shall denote the set of extremal functions satisfying equation (9), in which $Q(w)$ is a polynomial in $1/w$ of degree not higher than n. The term extremal can be justified by the fact that every such function gives a local extremum of some function-

nal; this is a rather deep fact, and we shall not need it. From equation (9) we obtain that on the circle $|z| = 1$ the function $f(z)$ can have only a finite number of algebraic singular points. Thus, $f(z)$ maps the disk $|z| < 1$ onto a domain whose boundary consists of a finite number of analytic arcs. On each such arc, by virtue of (9) and (18),

$$Q(f(z))(df(z)/f(z))^2 \geqslant 0,$$

i.e., these arcs belong to the trajectories of the differential $Q(w)(dw/w)^2$. Let us take the union of the trajectories of this differential that have limiting end points at the zeros of $Q(w)$. Adjoining to the set so constructed the point $w = 0$, we obtain a plane connected graph $\mathfrak{G}$—the graph of the differential $Q(w)(dw/w)^2$. The structure of the graph of a quadratic differential has been studied in detail in the papers ([3–5]). It is important to note that every cycle of the graph $\mathfrak{G}$ necessarily contains the point $w = 0$. It is not difficult to show that the image of the circle $|z| = 1$ under the mapping $w = f(z)$ will be a tree $\mathfrak{T} \subset \mathfrak{G}$. The point $w = \infty$ belongs to the tree $\mathfrak{T}$, and we shall regard it as the root of $\mathfrak{T}$.

Below we shall study the second variation in the class S_n. To each $f \in S_n$ there corresponds equation (9), or, what is the same, equation (7), which is completely determined by the vector $\lambda = (\lambda_1, \dots, \lambda_n)$. We shall call these vectors associated with f. The totality of all associated vectors will be denoted by Λ_f, and the totality of those vectors for which (18) holds will be denoted by K_e or $K_e(f)$; K_e is a cone. Note that $\dim K_e \geqslant 1$. A vector $\lambda \in K_e$ naturally determines the linear functional

$$L(f) = \operatorname{Re} \sum_1^n \overline{\lambda_k} a_k,$$

which plays an essential role in the class S_n. We transform formula (17) for the second variation of the linear functional $L(f)$. Let $D(\mathfrak{T})$ be the domain complementary to $\mathfrak{T}$. Put

$$\xi(w) = \delta f(z), \quad z = f^{-1}(w); \qquad \xi(w) = \sum_1^\infty \xi_k w^{k+1}.$$

The coefficients ξ_k, $k = 1, 2, \dots$, are easily computed in terms of δa_k, $k = 1, 2, \dots$. Let

$$Q(w) = \sum_1^n A_k w^{-k}.$$

Orient the tree $\mathfrak{T}$ in an arbitrary way. Then on each edge of the tree the sides—the left and the right—will be determined. We shall denote the limiting values of the function $\varphi(w)$ on $\mathfrak{T}$ from the left and from the right, respectively, by $\varphi_+(w)$, $\varphi_-(w)$. Put $\xi^*(w) = \sqrt{Q(w)/w^2}\,\xi(w)$, where some branch of the square root is taken. It turns out that the second variation is a quadratic functional of $\xi(w)$, i.e., of the first variation. Namely:

$$\delta^2 L(f) = \mathrm{Re}\left\{\frac{1}{2\pi i}\int_{\mathfrak{T}}[\xi_+^*(w)d\overline{\xi_+^*(w)} - \xi_-^*(w)d\overline{\xi_-^*(w)}] - \frac{1}{2}\sum_{k=1}^{n}\xi_k\sum_{l=1}^{n-k}(k+l+2)A_{k+l}\xi_l\right\}, \tag{19}$$

where the integral over $\mathfrak{T}$ is understood as the sum of integrals over the oriented edges. Duren and Schiffer, in the paper ([6]), assuming that the tree $\mathfrak{T}$ consists of one edge, established a formula close to our formula (19).

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Note: Figure translations are in progress. See original paper for figures.