ON THE THEORY OF SOLUTIONS OF NONLINEAR OPERATOR EQUATIONS

MATHEMATICS

1970

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Abstract

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UDC 517.948:513.88

MATHEMATICS

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ON THE THEORY OF SOLUTIONS OF NONLINEAR OPERATOR EQUATIONS

(Presented by Academician A. N. Tikhonov on 4 V 1970)

Let E be a Banach space. By E_T denote the space of continuous abstract functions $x(t)$ $(0 \le t \le T)$ with values in E and with norm

$$\|x(t)\| = \max_{0 \le t \le T} \|x(t)\|_E.$$

By S and S_T denote balls, respectively, in E and E_T, with centers at the point $x^{(0)}$ and radii r.

Let $F(t, x, y_t)$ be a nonlinear Volterra operator for fixed x (i.e., for fixed t it acts from E_t into E), and for fixed t and y_t an "ordinary" nonlinear operator (i.e., an operator acting in E).

The article is devoted to the study of solutions of the equation

$$x(t) = x^{(0)} + F[t, x(t), x_t] \qquad (1)$$

in E_T.

An example of equation (1) may be, for instance, the following integral equation of Uryson–Volterra type

$$x(t,s) = \Phi\left\{t, s; \int_0^T K_1[t, s, \sigma, x(t,\sigma)]\, d\sigma, \int_0^t K_2[t, s; \tau, x(\tau, s)]\, d\tau\right\}. \qquad (2)$$

In the case when $F(t, x, y_t)$ does not depend on x, equation (1) was first investigated by A. N. Tikhonov (^{1}). (Further investigations of such equations were considered, for example, in works ($^{2-5}$).) In the case when the operator $F(t, x, y_t)$ depends only on x, equation (1) is an "ordinary" nonlinear operator equation, for the solution of which a considerable number of fixed-point principles and approximate methods are known. Equation (1) in its general form is studied here for the first time.

By $\varphi(t, u_t)$ denote a scalar Volterra operator acting from M_T^+ into $(0, \infty)$ for fixed $t \in [0, T]$, where

$$M_T^+ = \left\{ 0 \leq u(t) \in C[0, T]; \ \max_{0 \leq t \leq T} u(t) \leq 2r \right\}.$$

Below we shall assume everywhere that if a monotone, bounded sequence $u^{(n)}(t)$ $(0 \leq t \leq T)$ converges pointwise to $u(t)$, then

$$\lim_{n \to \infty} \varphi(t, u_t^{(n)}) = \varphi(t, u_t).$$

1. First suppose that, for any fixed abstract function $\xi(t) \in S_T$, the operator equation

$$x(t) = x^{(0)} + F[t, x(t), \xi_t] \tag{3}$$

has a unique solution $x(t) \in S_T$, and it can be found.

For an approximate solution of equation (1), we construct successive approximations as follows:

$$x^{(0)}(t) = x^{(0)}, \tag{4}$$

$$x^{(n)}(t) = x^{(0)} + F\left[t, x^{(n)}(t), x_t^{(n-1)}\right] \qquad (n = 1, 2, ...).$$

Theorem 1. Suppose that the operator $F(t, x, y_t)$ is defined on the topological product $R = [0, T] \times S \times S_r$, is continuous, bounded, and satisfies the condition

$$\|F(t, \tilde{x}, \tilde{y}_t) - F(t, x, y_t)\| \leq L\|\tilde{x} - x\| + \varphi(t, \|\tilde{y}_t - y_t\|) \qquad (0 \leq L < 1),$$

where the Volterra operator $\varphi(t, u_t)$ $(0 \leq t \leq T, \ u \in M_T^+)$ is nondecreasing with respect to its second argument, the equation

$$u(t) = \frac{1}{1 - L} \varphi(t, u_t) \tag{5}$$

has only the zero solution, and

$$M = \max \left\{ \max_R \|F(t, x, y_t)\|, \ \frac{1}{1 - L} \max_{0 \leq t \leq T} \varphi(t, 2r) \right\} \leq 2r. \tag{6}$$

Then there exists a unique solution of equation (1), and it is the limit of the successive approximations (3), with the rate of convergence determined by the formula

$$\|x^{(n)}(t) - x(t)\| \le \varepsilon^{(n)}(t),$$

where $\{\varepsilon^{(n)}(t)\}$ is determined from the equalities

$$\varepsilon^{(0)}(t) = M,$$

$$\varepsilon^{(n)}(t) = \frac{1}{1-L}\varphi(t, \varepsilon_t^{(n-1)}) \qquad (n = 1, 2, \ldots).$$

Now, for an approximate solution of equation (1), we construct approximations as follows:

$$x^{(n)}(t) = x^{(0)} \qquad (0 \le t \le T/n),$$

$$x^{(n)}(t) = x^{(0)} + F\left[t - T/n, x^{(n)}(t), x_{t-T/n}^{(n)}\right] \qquad (T/n \le t \le T), \tag{7}$$

$$F[0, x(0), x_0] = 0 \qquad (x \in S_T).$$

Theorem 2. Suppose that the operator $F(t, x, y_t)$ is defined on the topological product R, is continuous, bounded, and satisfies the condition

$$\|F(t+\Delta t, \tilde{x}, \tilde{y}_{t+\Delta t}) - F(t, x, y_t)\| \le C|\Delta t| + L\|x - \tilde{x}\| + \varphi(t, \|\tilde{y}_t - y_t\|),$$

$$(0 \le L < 1, \qquad 0 \le t, \quad t + \Delta t \le T),$$

where the Volterra operator $\varphi(t, u_t)$ $(0 \le t \le T,\ u \in M_T^+)$ is nondecreasing in u_t, equation (5) has only the zero solution, and

a) for any fixed $\delta \in [0, CT/(1-L)]$ the equation

$$u(t) = \delta + \frac{1}{1-L}\varphi(t, u_t)$$

has a unique solution from M_T^+.

Then there exists a unique solution of equation (1), and it is the limit of approximations (7), with the rate of convergence determined by the formula

$$\|x^{(n)}(t) - x(t)\| \le \varepsilon^{(n)}(t),$$

where $\varepsilon^{(n)}(t)$ are solutions of the equations

$$u(t) = \frac{CT}{(1-L)n} + \frac{1}{1-L}\varphi(t, u_t).$$

2. Now suppose that we cannot find a solution of equation (3), but assume that, for any fixed abstract function $\xi \in S_T$, the Volterra equation

$$x(t) = x^{(0)} + F(t, \xi(t), x_t) \tag{8}$$

has a unique solution $x(t) \in S_T$, and that it can be found.

In this case we construct the successive approximations as follows:

$$x^{(0)}(t) = x^{(0)},$$

$$x^{(n)}(t) = x^{(0)} + F\left[t, x^{(n-1)}(t), x_t^{(n)}\right] \quad (n = 1, 2, ...). \tag{9}$$

Theorem 3. Let the operator $F(t, x, y_t)$ satisfy all the conditions of Theorem 1, with the only difference that condition (6) is replaced by the following condition.

For any fixed function $\alpha(t) \in M_T^+$, the scalar equation

$$u(t) = L\alpha(t) + \varphi(t, u_t)$$

has a unique solution $u(t) \in M_T^+$, and it can be found.

Then equation (1) has a unique solution, and it is the limit of the successive approximations (9); moreover, the rate of convergence is determined by the formula

$$\left\|x^{(n)}(t) - x(t)\right\| \le \varepsilon^{(n)}(t),$$

where $\varepsilon^{(n)}(t)$ are solutions of the equations

$$\varepsilon^{(0)}(t) = 2rL + \varphi(t, \varepsilon_t^{(0)}),$$

$$\varepsilon^{(n)}(t) = L\varepsilon^{(n-1)}(t) + \varphi(t, \varepsilon_t^{(n)}) \quad (n = 1, 2, ...).$$

3. In the preceding paragraphs we assumed that we can find a solution of equation (3) or (8). If this is impossible, then some approximate method may also be used to find solutions of these equations. For example, in this case the successive approximations for an approximate solution of equation (1) may be constructed by the equalities

$$x^{(0)}(t) = x^{(0)}, \tag{10}$$

$$x^{(n)}(t) = x^{(0)} + F\left[t, x^{(n-1)}(t), x_t^{(n-1)}\right] \quad (n = 1, 2, ...)$$

or

$$x^{(n)}(t) = x^{(0)} \quad (0 \leq t \leq T/n), \qquad x^{(0)}(t) = x^{(0)},$$

$$x^{(n)}(t) = x^{(0)} + F\left[t - T/n, x^{(n-1)}(t), x_{t-T/n}^{(n)}\right] \quad (T/n < t \leq T), \tag{11}$$

$$F[0, x(0), x_0] = 0 \quad (x \in S_T).$$

Theorem 4. Let the operator $F(t, x, y_t)$ satisfy all the conditions of Theorem 1. Suppose, in addition, that the condition

$$\max_R \|F(t, x, y_t)\| \leq r \tag{12}$$

is fulfilled.

Then equation (1) has a unique solution, and it is the limit of the successive approximations (10); moreover, the rate of convergence is determined by the formula

$$\left\|x^{(n)}(t) - x(t)\right\| \leq \varepsilon^{(n)}(t),$$

where $\varepsilon^{(n)}(t)$ is determined from the equalities

$$\varepsilon^{(0)}(t) = M,$$

$$\varepsilon^{(n)}(t) = L\varepsilon^{(n-1)}(t) + \varphi\left[t, \varepsilon_t^{(n-1)}\right] \quad (n = 1, 2, ...), \qquad \varepsilon^{(n)}(t) = M.$$

Theorem 5. Let the operator $F[t, x, y_t]$ satisfy all the conditions of Theorem 2, with only the difference that condition a) is replaced by condition b): for any fixed $\delta \in [0, CT]$ and for any fixed function $\alpha(t) \in M_T^+$, the scalar equation

$$u(t) = \delta + L\alpha(t) + \varphi(t, u_t)$$

has a unique solution $u(t) \in M_T^+$, and it can be found.

Suppose, moreover, that condition (12) is fulfilled.

Then equation (1) has a unique solution, and this solution is the limit of the successive approximations (11), with the rate of convergence determined by the formula

$$\|x^{(n)}(t) - x(t)\| \leq \varepsilon^{(n)}(t),$$

where $\varepsilon^{(n)}(t)$ are the solutions of the equations

$$\varepsilon^{(0)}(t) = 2rL + \varphi(t, \varepsilon_t^{(0)}) + CT,$$

$$\varepsilon^{(n)}(t) = L\varepsilon^{(n-1)}(t) + CT/n + \varphi(t, \varepsilon_t^{(n)}).$$

The proofs of all the theorems given above are based on theorems on Volterra operator inequalities (see, for example, [6]) and are carried out by the "majorant method" [7].

4. Let us note that the sequences $\varepsilon^{(n)}(t)$ appearing in the theorems given above converge uniformly to zero.

Let us also note that in all the theorems given above one can take, for example,

$$\varphi(t, u_t) \equiv \int_0^t L_0(s) u(s)\, ds,$$

where

$$L_0 = \int_0^t L_0(s)\, ds$$

is subject to certain conditions. In this case one can write an explicit expression for $\varepsilon^{(n)}(t)$.

Finally, let us note that the general theorems obtained can be applied, according to the standard scheme, to the investigation of equation (2).

Received
7 II 1970

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