ON SUMS OF CHARACTERS WITH PRIME NUMBERS

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Abstract

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ON SUMS OF CHARACTERS WITH PRIME NUMBERS

(Presented by Academician I. M. Vinogradov on 27 X 1969)

Notation: q is a sufficiently large prime number; χ is a nonprincipal character mod q; p is a prime number; k is an arbitrary integer not divisible by q; $\varepsilon > 0$ is arbitrarily small and not always the same.

In the paper the sums S_N are studied,

$$S_N = \sum_{p \le N} \chi(p + k). \tag{1}$$

Such sums were first studied by I. M. Vinogradov (see ([1–5])) with the aid of the method he created in 1934–1937 for estimating trigonometric sums, which made it possible to obtain a number of fundamental results in number theory.

The strongest result ([5]) is formulated as follows:

$$|S_N| \ll N^{1+\varepsilon}\Delta, \tag{2}$$

where $\Delta = q^{1/4}N^{-1/3} + N^{-1/10}$.

It is clear from (2) that the estimate will already be nontrivial for $N > q^{3/4+\varepsilon}$. We note that, if one assumes the generalized Riemann hypothesis, then from it one can obtain a nontrivial estimate of (1) only for $N > q^{1+\varepsilon}$. In the present work, by combining the method of I. M. Vinogradov with ideas from the author' s paper [6], a new estimate is obtained, nontrivial already for $N > q^{1/2+\varepsilon}$.

Theorem 1. Let ω be an arbitrary number in the interval $0 < \omega \le 1/4$, and let $q^{1/2+\omega} \le N \le q$. Then the estimate

$$|S_N| \ll Nq^{-\gamma\omega^2},$$

holds, where $\gamma > 0$ is an absolute constant, and the constant in $\ll$ depends only on ω.

We shall formulate three main lemmas needed for the proof of the theorem.

Lemma 1. Let γ_1 and γ_2 be positive numbers,

$$q^{\gamma_1} \leq M < M_1 \leq 2M < q^{1/2}, \qquad U > q^{1/2+\gamma_2},$$

$$T = \sum_{U<u\leq U}{}' \left| \sum_{M<m\leq \min(M_1, Nu^{-1})}{}' \chi(m + ku^{-1}) \right|,$$

where u, m run through values that are products, respectively, of c_1 and c_2 factors, each of which, independently of the others, runs through its own increasing sequence of positive integers. Then

$$T \ll UMq^{\varepsilon}\Delta,$$

where $\Delta = q^{0.5\gamma_1\gamma_2} + q^{-0.25\gamma_1}$.

Lemma 2. Let $\gamma_1, \gamma_2, \gamma_3$ be positive numbers,

$$0 < \gamma < 0.5(0.5\gamma_3 - \gamma_2).$$

$$q^{\gamma_1} \ll M \ll q^{1/4+\gamma_2}, \qquad M < M_1 \leq 2M, \qquad q^{1/2+\gamma_3} \ll UM \leq q,$$

$$T = \sum_{U<u\leq 2U}{}' \left| \sum_{M<m\leq \min(M_1, Nu^{-1})} \chi(m + ku^{-1}) \right|,$$

where u runs through values that are products of factors each of which, independently of the others, runs through its own increasing sequence of positive integers, and m runs through consecutive integer values of the interval $(M, \min(M_1, Nu^{-1})]$. Then

$$T \ll UMq^{\varepsilon}\Delta,$$

where

$$\Delta = q^{-\gamma_1\left(1 - \frac{1}{1+0.5\gamma_3-\gamma_2-2\gamma_1}\right)}.$$

Lemma 3. Let $\gamma > 0$, M be an integer,

$$q^{1/4+\gamma} \ll M \ll q^{5/8}, \qquad 0 \le a \le q-1, \qquad T = \sum_{m=1}^{M} \chi(m+a).$$

Then

$$T \ll M q^{\varepsilon} \Delta,$$

where $\Delta = q^{-0.5\gamma^2}$.

The proof of Theorem 1 is carried out by combining the method of I. M. Vinogradov (see, for example, the proof of the theorem in [3]) with the results of Lemmas 1, 2, 3.

From Theorem 1 one rather easily obtains new theorems on the distribution of residues and nonresidues mod q of arbitrary degree in sequences of the form $p+k,\ p \le N$.

Theorem 2. Let $n \mid (q-1)$, $2 \le n \le q-2$, and $0 \le s \le n-1$. Then, under the conditions of Theorem 1, for the number T_s of those numbers $p+k$ which satisfy the conditions

$$p \le N, \qquad \operatorname{ind}(p+k) \equiv s \pmod n,$$

the formula

$$T_s = \frac{1}{n}\pi(N) + O(Nq^{-\gamma\omega^2})$$

holds.

Theorem 3. Under the conditions of Theorem 1, the number of quadratic residues and the number of quadratic nonresidues mod q of the form $p+k,\ p \le N$, is equal to

$$\frac{1}{2}\pi(N) + O(Nq^{-\gamma\omega^2}).$$

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Note: Figure translations are in progress. See original paper for figures.