

# ON THE GREEN' S FUNCTION OF THE STURM-LIOUVILLE EQUATION WITH OPERATOR COEFFICIENTS

MATHEMATICS

1970

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.64399>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 517.43+513.881

**MATHEMATICS**

**E. ABDUKADYROV**

**ON THE GREEN'S FUNCTION OF THE STURM-LIOUVILLE EQUATION WITH OPERATOR COEFFICIENTS**

*(Presented by Academician A. N. Tikhonov on 21 IV 1970)*

1. Let  $H$  be a separable Hilbert space. Consider the set of all Bochner-measurable functions  $f(x)$  ( $-\infty < x < \infty$ ) with values in  $H$ , for which

$$\int_{-\infty}^{\infty} \|f(x)\|_H^2 dx < \infty.$$

This set of functions forms a new Hilbert space  $H_1$  (also separable), if the scalar product in it is defined by the equality

$$(f(x), g(x))_{H_1} = \int_{-\infty}^{\infty} (f(x), g(x))_H dx.$$

Let  $P(x)$  and  $Q(x)$  ( $-\infty < x < \infty$ ) be families of operators in  $H$ . In the space  $H_1$  consider the differential expression

$$l(y) = -(P(x)y')' + Q(x)y, \tag{1}$$

defined on some set of functions  $D \subset H_1$  (derivatives are understood in the strong sense).

It can be shown that, under certain restrictions on the families of operators  $P(x)$  and  $Q(x)$ , the differential expression  $l$  can be extended to a self-adjoint operator  $\bar{L}$  in  $H_1$ .

In the work of B. M. Levitan <sup>(1)</sup>, the operator Green's function of the operator  $\bar{L}$  was studied for  $P(x) = E$ , where  $E$  is the identity operator in the space  $H$ . The principal aim of the present note is the study of the operator Green's function of the operator  $\bar{L}$  for  $P(x) \neq E$ .

2. We list the main restrictions on the operators  $P(x)$  and  $Q(x)$  under which we have been able to study the operator Green's function of the operator  $\bar{L}$ :

1) For all  $x$  ( $-\infty < x < \infty$ ) and all  $f \in H$ ,

$$m(f, f)_H \leq (P(x)f, f)_H \leq M(f, f)_H,$$

where  $m$  and  $M$  are constant positive numbers.

2) The operator function  $P(x)$  ( $-\infty < x < \infty$ ) is everywhere uniformly differentiable.

3) The operators  $Q(x)$  are self-adjoint in  $H$  for almost all  $x$ , and for all (almost all)  $x$  there exists a common everywhere dense in  $H$  set  $D\{Q(x)\}$ , on which the  $Q(x)$  are defined and symmetric\*.

4) The operators  $Q(x)$  are uniformly bounded from below; that is, there exists a constant number  $d < \infty$  such that for almost all  $x$  and all  $f \in D\{Q(x)\}$ ,

$$(Q(x)f, f)_H \geq -d(f, f)_H.$$

5) There exist constant numbers  $A > 0$ ,  $0 < a < 3/2$  such that for all  $x$  and  $|x - \xi| \leq 1$  the inequality

$$\|[Q(\xi) - Q(x)]Q^{-a}(x)\|_H < A|x - \xi|$$

holds.

6) For  $|x - \xi| > 1$ ,

$$\|K(\xi) \exp\{-\frac{1}{2}|x - \xi|\omega\}\|_H < B,$$

where  $\omega = \{K(x) + \mu P^{-1}(x)\}^{1/2}$ ,  $K(x) = P^{-1/2}(x)Q(x)P^{-1/2}(x)$ , and  $B$  is a constant number.

\* Thus, we admit that the operators  $Q(x)$  may be unbounded in  $H$ .

7) For all  $x$  and  $\xi$  from  $(-\infty, \infty)$

$$\|Q(x)P^{\pm 1/2}(x)Q^{-1}(x)\|_H < B_1, \quad \|Q(\xi)P^{-1/2}(x)P^{1/2}(\xi)Q^{-1}(\xi)\|_H < B_2,$$

where  $B_1, B_2$  are constants.

Suppose that  $Q(x)$ , for almost all  $x$ , is the inverse of a completely continuous operator. Then  $K(x)$ , too, for almost all  $x$ , is the inverse of a completely continuous operator. Denote by  $\beta_1(x) \leq \beta_2(x) \leq \beta_3(x) \leq \dots$  the eigenvalues

of the operator  $K(x)$ , with respect to which we shall assume that they are measurable functions. Without loss of generality, one may assume that  $\beta_1(x) \geq 1$ .

8) The series

$$\sum_{i=1}^{\infty} \beta_i^{-3/2}(x)$$

converges and its sum  $F(x)$  is a function of the class

$$L_1(-\infty, \infty), \quad \text{i.e.} \quad \int_{-\infty}^{\infty} F(x) dx < \infty.$$

3. If conditions 1), 3), and 4) are satisfied, there exists a Friedrichs self-adjoint extension of the operator  $l$ . We denote it by  $L$ . The operator  $L$  in  $H_1$  is bounded below by the same number  $d$ . Therefore, if  $\mu > d$ , then there exists the inverse operator  $(L + \mu E)^{-1} = R_\mu$ . The following holds.

**Theorem 1.** If conditions 1)–8) are fulfilled, then the operator  $R_\mu$ , for all  $\mu > d$ , is an integral operator with operator kernel  $G(x, \eta; \mu)$ , which is an operator function in  $H$ , depending on two variables  $x$  and  $\eta$  ( $-\infty < x, \eta < \infty$ ) and on the parameter  $\mu$ , and satisfies the following conditions:

- a)  $G(x, \eta; \mu)$  is strongly continuous in the variables  $(x, \eta)$ ;
- b) there exists a strong derivative  $\partial G / \partial \eta$ , and

$$\frac{\partial G}{\partial \eta}(x, x + 0; \mu) - \frac{\partial G}{\partial \eta}(x, x - 0; \mu) = -P^{-1}(x),$$

the equality being understood in the sense of equality (0.5) of paper (1);

c)

$$-(G'_\eta P(\eta))'_\eta + G\{Q(\eta) + \mu E\} = 0,$$

the equality being understood in the sense of equality of item c) of paper (1);

d)  $G^*(x, \eta; \mu) = G(\eta, x; \mu)$ .

Here the asterisk denotes the adjoint operator in  $H$ .

4. The theorem is proved by the method of paper (1), using some results from (2) and the following lemmas.

Put

$$g(x, \eta; \mu) = \frac{1}{2} P^{-1/2}(x) \omega^{-1} e^{-|x-\eta|\omega} P^{-1/2}(x)$$

and consider the integral equation

$$\begin{aligned} G(x, \eta; \mu) = & g(x, \eta; \mu) - \int_{-\infty}^{\infty} g(x, \xi; \mu) [Q(\xi) - Q(x)] G(\xi, \eta; \mu) d\xi + \\ & + \int_{-\infty}^{\infty} \left[ \frac{1}{2} P^{-1/2}(x) \omega \exp(-|x - \xi|\omega) P^{-1/2}(x) (P(\xi) - P(x)) + g'_\xi(x, \xi; \mu) P'(\xi) \right] \times \\ & \times G(\xi, \eta; \mu) d\xi = g(x, \eta; \mu) - NG. \end{aligned} \quad (2)$$

Consider the Banach spaces  $X_1, X_2, X_3^{(p)}$  ( $p \geq 1$ ),  $X_1^{(s)}, X_2^{(s)}, X_4^{(s)}$  ( $s \leq 0$ ), whose elements are operator functions  $A(x, s)$  in  $H$  ( $-\infty < x, s < \infty$ ) (the definitions of these spaces are given in paper (1)).

**Lemma 1.** If the operator functions  $P(x)$  and  $Q(x)$  satisfy conditions 1)–7), then for sufficiently large  $\mu$  ( $\mu > \mu_0$ ) the operator  $N$  is a contraction operator in the spaces  $X_1, X_2, X_1^{(s)}, X_2^{(s)}$ , and  $X_4^{(s)}$ .

In all the Banach spaces considered above, equation (2) has a unique solution, which can be obtained by iteration if and only if the operator function  $g(x, \eta; \mu)$  belongs to the corresponding space. It is easy to verify that  $g(x, \eta; \mu) \in X_3 \equiv X_3^{(1)}$ . Therefore  $G(x, \eta; \mu)$ , for sufficiently large  $\mu$  ( $\mu > \mu_0$ ), also belongs to  $X_3$ , provided only that the operator functions  $P(x)$  and  $Q(x)$  satisfy conditions 1)–7). If, in addition to these conditions, condition 8) also holds, then  $g(x, \eta; \mu) \in X_2$ , and, consequently,  $G(x, \eta; \mu)$ , for sufficiently large  $\mu$  ( $\mu > \mu_0$ ), is also an element of the space  $X_2$ .

**Lemma 2.** Let the operator functions  $P(x)$  and  $Q(x)$  satisfy conditions 1), 3), 4), 6) and 7), and the condition: for  $|x - \eta| \leq 1$ ,

$$\|Q^{-1/2}(x)Q^{1/2}(\eta)\|_H \leq C,$$

where  $C$  is a constant.

Then the operator function  $g(x, \eta; \mu)$  belongs to the space  $X_4^{(1/2)}$ , i.e.

$$\sup_{-\infty < x < \infty} \int_{-\infty}^{\infty} \|g(x, \eta; \mu)Q^{1/2}(\eta)\|_H d\eta < \infty.$$

If  $Q(x)$  also satisfies conditions 2), 5), then  $G(x, \eta; \mu)$ , for sufficiently large  $\mu > \mu_0$ , also belongs to the space  $X_4^{(1/2)}$ .

**Lemma 3.** Let the operator functions  $P(x)$  and  $Q(x)$  satisfy conditions 1), 3), 4), 6) and 7), and the condition: for  $|x - \eta| \leq 1$ ,

$$\|Q^{1/2}(x)Q^{-1/2}(\eta)\|_H \leq C.$$

Then the operator function  $\partial^2 g / \partial \eta^2$  ( $\eta \neq x$ ) belongs to  $X_4^{(-1/2)}$ , i.e.

$$\sup_{-\infty < x < \infty} \int_{-\infty}^{\infty} \left\| \frac{\partial^2 g}{\partial \eta^2} Q^{-1/2}(\eta) \right\|_H d\eta < \infty.$$

Now let  $\mu' \in (d, \mu_0]$ . Consider the integral equation

$$G(x, \eta; \mu') = G(x, \eta; \mu) + (\mu - \mu') \int_{-\infty}^{\infty} G(x, s; \mu) G(s, \eta; \mu') ds, \quad (3)$$

where  $G(x, s; \mu)$  is the Green operator function of the operator  $L$ , constructed above, for  $\mu > \mu_0$ . By the preceding,

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \|G(x, s; \mu)\|_2^2 ds \right\} dx < \infty. \quad (4)$$

Here  $\|G(x, s; \mu)\|_2$  is the Hilbert–Schmidt norm of the operator  $G(x, s; \mu)$  in  $H$  for fixed  $(x, s; \mu)$ . By virtue of (4) (see <sup>(1, 3)</sup>) the integral operator

$$Af = \int_{-\infty}^{\infty} G(x, s; \mu) f(s) ds, \quad f(s) \in H_1$$

is completely continuous in  $H_1$ . Since  $\mu - \mu'$  is a resolvent point of the operator  $A$  (see <sup>(4)</sup>), the integral equation (3) has a unique solution  $G(x, \eta; \mu')$ . Splitting the integral operator in (3) into a finite-dimensional operator and an integral operator whose norm is sufficiently small, one can show that the operator  $G(x, \eta; \mu')$  is uniformly bounded in  $H$ . By virtue of the properties of the Green operator function  $G(x, \eta; \mu)$ , one can prove that  $G(x, \eta; \mu')$  is the (unique) Green operator function of the operator  $L$  for  $\mu' \in (d, \mu_0)$ .

5. Analogous results can be obtained in the case of the half-line  $(0, \infty)$ . Let the operator functions  $P(x)$  and  $Q(x)$  be defined on the half-line  $(0, \infty)$  and satisfy conditions 1)–8). Consider the operator

$$l(y) = -(P(x)y')' + Q(x)y, \quad (0 < x < \infty) \quad (5)$$

and adjoin to it the boundary condition at zero

$$y'(0) = hy(0), \quad (6)$$

where  $h$  is a constant self-adjoint operator in  $H$  (possibly unbounded).

Let us show how the analogue of the function  $g(x, \eta; \mu)$  is constructed. Put

$$K(x) = P^{-1/2}(x)Q(x)P^{-1/2}(x), \quad \omega = \{K(x) + \mu P^{-1}(x)\}^{1/2},$$

$$X(\eta) = P^{-1/2}(x)[\operatorname{ch} \omega \eta + h\omega^{-1} \operatorname{sh} \omega \eta]P^{-1/2}(x) \quad (0 < x, \eta < \infty).$$

Then

$$g(x, \eta; \mu) = \begin{cases} X(\eta)P^{1/2}(x)(\omega + h)^{-1}e^{-\omega x}P^{-1/2}(x), & \text{if } \eta \leq x, \\ P^{-1/2}(x)e^{-\omega \eta}(\omega + h)^{-1}P^{1/2}(x)X(x), & \text{if } \eta > x. \end{cases}$$

It is easy to verify that the function  $g(x, \eta; \mu)$  satisfies the conditions:

$$P(x) \frac{\partial^2 g}{\partial \eta^2} = \{Q(x) + \mu E\}g, \quad \left( \frac{\partial g}{\partial \eta} - hg \right)_{\eta=0} = 0,$$

$$\frac{\partial g}{\partial \eta} \Big|_{\eta=x+0} - \frac{\partial g}{\partial \eta} \Big|_{\eta=x-0} = -P^{-1}(x).$$

Having constructed the function  $g(x, \eta; \mu)$ , one can set up an integral equation analogous to equation (2) (the limits of integration will now be 0 and  $\infty$ ), and then prove the existence of an operator function  $G(x, \eta; \mu)$ , which is the kernel of the operator  $(L + \mu E)^{-1}$ , where  $L$  is the Friedrichs extension of the operator (5) + (6).

The operator Green's function on a finite interval is studied analogously. We note that a problem of this type was considered in the works <sup>(4,5)</sup>.

6. As an example, consider the eigenvalue problem for an operator of the form

$$-\sum_{i=1}^3 \frac{\partial}{\partial x_i} \left( a_i(x) \frac{\partial}{\partial x_i} u(x) \right) + q(x)u(x) = \lambda u(x), \quad (7)$$

$$u|_{\Gamma} = 0 \quad (8)$$

in the cylindrical domain

$$\Pi = \Pi((x_2, x_3) \in \Omega; -\infty < x_1 < \infty), \quad x = (x_1, x_2, x_3),$$

where  $\Omega$  is a plane simply connected domain of the  $(x_2, x_3)$ -plane, and  $\Gamma$  is the boundary of the cylinder  $\Pi$ . The coefficients  $a_i(x)$  are positive, uniformly

bounded functions on  $\Pi$ , and  $q(x) \rightarrow +\infty$  as  $|x| \rightarrow +\infty$  (by another method, for more general domains, the analogous problem with  $a_i(x) \equiv 1$  ( $i = 1, 2, 3$ ) was studied in a paper by A. G. Ramm <sup>(6)</sup>). Equality (7) can be rewritten in operator form (in the present case  $H = L_2(\Omega)$ ,  $H_1 = L_2(\Omega, -\infty < x_1 < \infty)$ )

$$-\frac{d}{dx_1} \left( P(x_1) \frac{du}{dx_1} \right) + \{Q(x_1) - \lambda E\}u = 0,$$

where

$$Q(x_1)u = -\frac{\partial}{\partial x_2} \left( a_2(x) \frac{\partial u}{\partial x_2} \right) - \frac{\partial}{\partial x_3} \left( a_3(x) \frac{\partial u}{\partial x_3} \right) + q(x)u, \quad P(x_1) \equiv a_1(x)E.$$

Applying the results set forth above to this case, one can study the properties of the Green's function of problem (7) + (8) and derive an asymptotic formula for the number of eigenvalues of this problem.

In conclusion, I express my deep gratitude to B. M. Levitan for posing the problem and for his great attention to the work.

Moscow State University  
named after M. V. Lomonosov

Received  
17 IV 1970

## REFERENCES

1. B. M. Levitan, *Matem. sborn.*, **76**, No. 2, 239 (1968).
2. S. G. Krein, *Linear Differential Equations in Banach Space*, "Nauka," 1967.
3. G. I. Laptev, *Differential Equations*, **2**, No. 9, 1151 (1966).
4. G. I. Laptev, *Lithuanian Mathematical Collection*, **8**, No. 1, 87 (1968).
5. S. G. Krein, G. I. Laptev, *Differential Equations*, **2**, No. 7, 919 (1966).
6. A. G. Ramm, *DAN*, **183**, No. 4, 780 (1968).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*