ON LOCAL CONJUGACY OF DIFFEOMORPHISMS

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ON LOCAL CONJUGACY OF DIFFEOMORPHISMS

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In the present note we study conditions for the conjugacy of local* diffeomorphisms in R^n of class C^m $(m \leq \infty)$ that leave the point zero fixed. This question was considered by S. Sternberg in ($^{1-3}$). Sternberg' s method is based on rather complicated iteration procedures. It is natural to try to replace the iterations by a suitable fixed-point principle. We shall show that such an approach not only simplifies the matter, but also leads to somewhat more precise results. We shall distinguish the cases $m = \infty$ and $m < \infty$. In the case $m = \infty$, it is possible to achieve considerably greater clarity in the exposition by using a natural algebraic language.

Let F and G be two local diffeomorphisms in R^n of class C^∞ leaving zero fixed. We shall say that F and G are conjugate if there exists a local diffeomorphism Φ of class C^∞ such that $F(\Phi(x)) = \Phi(G(x))$ in some neighborhood of zero. Obviously, for conjugacy it is first of all necessary that there be formal conjugacy, i.e., solvability of the infinite system of algebraic equations

$$(F(\Phi))^{(\nu)}\big|_{x=0} = (\Phi(G))^{(\nu)}\big|_{x=0} \quad (\nu = 0, 1, 2, \ldots) \tag{1}$$

with respect to the Taylor coefficients of the mapping Φ (here and below $F^{(\nu)}(x)$ denotes the total derivative of order ν of the mapping F). It is easy to give an example where formal conjugacy does not imply conjugacy of diffeomorphisms. Namely, put $n = 1$, $F(x) = x$, $G(x) = x + \omega(x)$, where $\omega^{(\nu)}(0) = 0$, $\nu = 0, 1, 2, \ldots$, $\omega \not\equiv 0$. In this case, obviously, system (1) is solvable, but F and G are not conjugate.

Theorem 1. *If the matrix $\Lambda = F'(0)$ has no spectrum on the unit circle, then for conjugacy of the diffeomorphisms F and G their formal conjugacy is necessary and sufficient.*

Let now K be the ring of germs of infinitely differentiable functions at the point $x = 0$, vanishing at this point, endowed with the usual topology. Every local diffeomorphism F generates a continuous automorphism U_F of the ring K:

$(U_F\tau)(x) = \tau(F(x))$. Conversely, every continuous automorphism of the ring K is generated, in the indicated sense, by some local diffeomorphism. Since the equality $F(\Phi(x)) = \Phi(G(x))$ is equivalent to the equality $U_\Phi U_F = U_G U_\Phi$, the question of conjugacy of the diffeomorphisms F and G is equivalent to the question of conjugacy of the automorphisms U_F and U_G.

Let $I \subset K$ be some ideal. We agree to denote by $\tilde{I}$ the ideal of germs all of whose partial derivatives lie in I. In particular, $\widetilde{K}$ is the ideal of germs of functions having at the point $x = 0$ a zero of infinite order. Denote by φ the homomorphism of the ring K into the ring of formal power series that assigns to each germ from K its formal Taylor series at the point $x = 0$. The kernel of this homomorphism is, obviously, the ideal $\widetilde{K}$. According to the well-known Borel theorem, the homomorphism φ is an epimorphism onto the ring of formal power series. The ideal

* That is, acting in a neighborhood of zero.

$\widetilde{K}$ is, obviously, invariant with respect to all automorphisms of the ring K. Consequently, each automorphism U of the ring K induces a certain automorphism $\widehat{U}$ of the factor $K/\widetilde{K}$. The condition of formal conjugacy of the diffeomorphisms F and G is equivalent to the condition of conjugacy of the induced automorphisms $\widehat{U}_F, \widehat{U}_G$. Therefore Theorem 1 can be formulated as follows:

Theorem 1′. *If the matrix $\Lambda = F'(0)$ has no spectrum on the unit circle, then for the conjugacy of the diffeomorphisms F and G it is necessary and sufficient that the induced automorphisms $\widehat{U}_F$ and $\widehat{U}_G$ be conjugate.*

We pass to the exposition of the proof of Theorem 1. Every diffeomorphism F can be written in the form

$$F(x) = \Lambda x + f(x), \qquad f(x) = O(\|x\|^2).$$

From the formal conjugacy of the diffeomorphisms F and G it follows that the matrices $F'(0)$ and $G'(0)$ are similar, i.e., one may assume that

$$G(x) = \Lambda x + g(x), \qquad g(x) = O(\|x\|^2).$$

Let* L_+, L_0, L_- be the invariant subspaces of the matrix Λ, corresponding respectively to the spectrum inside, outside, and on the unit circle. One may assume the basis in R^n chosen in such a way that if $x = (x_+, x_0, x_-)$, then $L_+ = \{(x_+, 0, 0)\}$, $L_0 = \{(0, x_0, 0)\}$, $L_- = \{(0, 0, x_-)\}$. We shall denote further by I_+, I_0, I_- the ideals of germs equal to zero on $L_0 + L_-$, $L_+ + L_-$, $L_0 + L_+$, respectively. Obviously, $K = I_+ + I_0 + I_-$.

Lemma 1. *Let the ideals I_+, I_0, I_- be invariant with respect to the automorphisms U_F, U_G. Then from the conjugacy of the automorphisms induced by U_F*

and U_G on the factor-ring[*] $K/\overline{I_+ + I_-}$, there follows the conjugacy of U_F and U_G.*

Since $\widetilde{K} \supset \overline{I_+ + I_-}$, the condition of the lemma entails the conjugacy of the automorphisms $\widehat{U}_F$ and $\widehat{U}_G$, induced on the factor $K/\widetilde{K}$. If $I_0 = 0$, then $\overline{I_+ + I_-} = \widetilde{K}$. Therefore the following holds.

Corollary. *If the matrix Λ has no spectrum on the unit circle and if the ideals I_+, I_- are invariant with respect to U_F, U_G, then the conjugacy of the induced automorphisms $\widehat{U}_F, \widehat{U}_G$ entails the conjugacy of the diffeomorphisms F and G.*

Theorem 1 follows from Lemma 1 and the known Hadamard–Perron theorem on invariant manifolds, which it is convenient for us to formulate in the following form:

Lemma 2. *If the matrix $\Lambda = F'(0)$ has no spectrum on the unit circle, then the automorphism U_F is conjugate to some automorphism U_H, for which the ideals I_+ and I_- are invariant.*

Both lemmas can be proved by applying the fixed-point principle. We give the proof of Lemma 1, dividing it into two parts.

1. Conjugacy modulo $\overline{I_+ + I_-}$ entails conjugacy modulo I_-. Indeed, let $\Phi(x) = x + \varphi(x)$. Then the conjugacy equation has the form:

$$\varphi(x) = \Lambda^{-1}\varphi(F(x)) + \Lambda^{-1}[f(x) - g(x + \varphi(x))] \equiv A\varphi. \qquad (2)$$

Since F and G are conjugate modulo $\overline{I_+ + I_-}$, one may assume that f and g have identical derivatives for $x \in L_0$. Therefore the operator $A\varphi$

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[*] Here and in Lemma 1, Λ may have spectrum on the unit circle.

[*]* In what follows we shall say more briefly: the diffeomorphisms F and G are conjugate modulo the ideal $\overline{I_+ + I_-}$.

maps into itself the space of mappings $\varphi \in C^\infty$ for which all derivatives are equal to zero for $x \in L_0$. Further, conjugacy modulo $\widetilde{I}_-$ is equivalent to the existence of such a diffeomorphism $\Phi(x) = x + \varphi(x)$ that

$$(\varphi(x))^{(\nu)}\big|_{x \in L_+ + L_0} = (A\varphi)^{(\nu)}\big|_{x \in L_+ + L_0}.$$

To construct such a diffeomorphism, by the well-known Whitney extension theorem, it suffices to find the values $(\varphi(x))^{(\nu)}\big|_{x \in L_+ + L_0}$. Fix a neighborhood $W \subset L_+ + L_0$ and consider the equation

$$\varphi(x) = \Lambda^{-1}\varphi(\Lambda_+x_+ + f_+(x),\ P(\Lambda_0x_0 + f_0(x)), \Lambda_-x_- + f_-(x)) + \Lambda^{-1}[f(x) - g(x + \varphi(x))], \tag{3}$$

where $f : L_0 \to W \cap L_0$ is a C^∞-mapping coinciding with the identity in a small neighborhood of zero. It is clear that if φ satisfies (3), then in a small neighborhood of zero it satisfies (2). Put in (3) $x_- = 0$ and $\hat{x} = (x_+, x_0, 0)$. Then we obtain the equation

$$\varphi(\hat{x}) = \Lambda^{-1}\varphi(\Lambda_+x_+ + f_+(\hat{x}), P(\Lambda_0x_0 + f_0(\hat{x})), 0) + \Lambda^{-1}[f(\hat{x}) - g(\hat{x} + \varphi(\hat{x}))]. \tag{4}$$

A solution of (4) may be sought in the space of mappings φ having, with respect to the variable x_+, a zero of infinite order, i.e.

$$\|\varphi^{(\nu)}(\hat{x})\| \leq c_{\nu j}\|x_+\|^j, \quad \hat{x} \in W, \quad j = N, N+1, ...;\ \nu = 1, 2, ... \tag{5}$$

For fixed $c_{\nu j}$ and $\hat{N} = N(\nu)$, the system of inequalities (5) defines a convex compact set in $C^\infty(W)$. Since Λ_+ is a contraction, for a suitable choice of the constants $c_{\nu j}$ and $N(\nu)$ this compact set is invariant with respect to the operator on the right-hand side of (4). By the well-known fixed-point principle, (4) has a solution $\varphi(\hat{x}) = \varphi(x_+, x_0, 0)$ of equation (4). Differentiating now (3) and putting $x_- = 0$, we obtain an equation for finding the derivative $\varphi^{(\nu)}(x)\big|_{x=L_+ + L_0}$, $\nu = 1, 2, ...$, which is solved in the same way. Thus, 1 is proved. Passing to the inverse mapping, we obtain that conjugacy modulo $\tilde{I}_+ + \tilde{I}_-$ entails conjugacy modulo $\tilde{I}_+$.

2. From conjugacy modulo $\tilde{I}_+$ (or $\tilde{I}_-$) follows conjugacy on all of K. Since F and G are conjugate modulo $\tilde{I}_+$, one may assume that f and g have identical derivatives for $x \in L_- + L_0$. In other words, the operator A in (2) maps into itself the space of mappings φ for which all derivatives are equal to zero for $x \in L_- + L_0$. Fix a neighborhood $V \subset R^n$ and consider the equation

$$\varphi(x) = \Lambda^{-1}\varphi(F_+(x), P(F_0(x), F_-(x))) + \Lambda^{-1}[f(x) - g(\Phi(x))], \quad x \in V, \tag{6}$$

where $P : L_- + L_0 \to V \cap L_- + L_0$ is a C^∞-mapping coinciding with the identity in a small neighborhood of zero. If φ satisfies (6), then in a small neighborhood of zero it satisfies (2). A solution of (6) may be sought in the space of mappings having a zero of infinite order with respect to the variable x_+:

$$\|\varphi^{(\nu)}(x)\| \leq c_{\nu j}\|x_+\|^j, \quad x \in V, \quad j = N, N+1, ...;\ \nu = 1, 2, ...$$

This system of inequalities describes a convex compact set in $C^\infty(V)$, which, for a suitable choice of $c_{\nu j}$ and $N(\nu)$, is invariant with respect to the operator on the right-hand side of (6). Consequently, (6) has a solution. The lemma is proved.

We now formulate without proof the results pertaining to the situation of finite smoothness. These results are obtained, in essence, by the same method, but with the corresponding technical conditions.

Let $\rho_1 > \rho_2 > \dots > \rho_s$ be the moduli of the eigenvalues of the matrix Λ, and let L_i be the invariant subspace corresponding to the part of the spectrum lying on the circle $|\lambda| = \rho_i$. Further, let $\xi_{i1}, \xi_{i2}, \dots, \xi_{in}$ be coordinates in L_i. Fix a multi-index $g = g_1 g_2 \cdots g_s$ ($g_i \geqslant 1$) and denote by $C^{g,\alpha}$ the class of diffeomorphisms F having partial derivatives up to

$$\frac{\partial^{|g|} F}{\partial \xi_{11}^{\gamma_{11}} \cdots \partial \xi_{1n_1}^{\gamma_{1n_1}} \cdots \partial \xi_{s1}^{\gamma_{s1}} \cdots \partial \xi_{sn_s}^{\gamma_{sn_s}}}, \qquad \sum_{k=1}^{n_i} \gamma_{ik} = g_i, \tag{7}$$

inclusive, where the derivatives (7) belong to $\operatorname{Lip} \alpha$.

Let F be a local diffeomorphism of class $C^{g,1}$, and let the matrix $\Lambda = F'(0)$ have no spectrum on the unit circle. Suppose, moreover, that the system of equations

$$\left. \frac{\partial^t (\Phi F)}{\partial \xi_{11}^{\delta_{11}} \cdots \partial \xi_{sn_s}^{\delta_{sn_s}}} \right|_{x=0} = \left. \frac{\partial^t (\Lambda \Phi)}{\partial \xi_{11}^{\delta_{11}} \cdots \partial \xi_{sn_s}^{\delta_{sn_s}}} \right|_{x=0}$$

has a solution for all partial derivatives up to (7) inclusive with respect to the unknown diffeomorphism $\Phi(x) = x + \varphi(x)$. Under these conditions the following holds.

Theorem 2. There exists a diffeomorphism $\Phi \in C^{g,\alpha}$, for some $\alpha < 1$, such that $\Phi(F(x)) = \Lambda(\Phi(x))$ in some neighborhood of the point $x = 0$.

Let now $F \in C^m$ ($m \geqslant 1$ an integer), let the matrix $\Lambda = F'(0)$ have no spectrum on the unit circle, and let the system of equations

$$(\Phi(F))^{(\nu)}\big|_{x=0} = (\Lambda \Phi)^{(\nu)}\big|_{x=0}, \qquad \nu = 1, 2, \dots, m,$$

be solvable with respect to the unknown mapping $\Phi(x) = x + \varphi(x)$. Under these conditions, Theorem 2 implies

Theorem 3. There exists a diffeomorphism $\Phi \in C^{[m/n]-1}$ such that $\Phi(F(x)) = \Lambda(\Phi(x))$ in some neighborhood of the point $x = 0$.

We note that in Sh. Sternberg the smoothness class depended on Λ. In the analogous theorem on topological conjugacy only the dichotomy condition remains: there must be no points of the spectrum on the unit circle.

In conclusion, we note that by the same method one can prove conjugacy theorems in a subgroup of the group of diffeomorphisms; for example, the theorem on volume-preserving mappings appearing in (5).

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