INFINITE NONABELIAN GROUPS WITH AN INVARIANCE CONDITION FOR INFINITE NONABELIAN SUBGROUPS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 519.41/47

MATHEMATICS

S. N. CHERNIKOV

INFINITE NONABELIAN GROUPS WITH AN INVARIANCE CONDITION FOR INFINITE NONABELIAN SUBGROUPS

(Presented by Academician V. M. Glushkov, 30 III 1970)

Infinite nonabelian groups all of whose infinite nonabelian subgroups are normal divisors (invariant subgroups) will be called $\overline{IN}$-groups, taking into account the name IN-groups, introduced in ([1]) for infinite nonabelian groups in which the infinite abelian subgroups are invariant. The class of $\overline{IN}$-groups contains, in particular, infinite nonabelian groups in which all nonabelian subgroups are invariant; arbitrary nonabelian groups (both infinite and finite) with this condition were studied in ([2,3]) and there received the name of metahamiltonian groups. The class of $\overline{IN}$-groups contains, in particular, also infinite nonabelian groups with the condition of invariance for all infinite subgroups, studied in ([4a]) (and there given the name INH-groups), as well as infinite nonabelian groups having no proper nonabelian subgroups, considered in ([4]). It also contains, obviously, infinite nonabelian groups having no proper infinite subgroups (groups defined in this way are called below Schmidt groups). The groups introduced by the definition of $\overline{IN}$-groups proposed here may, of course, be regarded as one of the possible generalizations of Schmidt groups.

As is known, the question of the existence of Schmidt groups has still not been settled; it is the fundamental question of the problem posed by O. Yu. Schmidt in 1938 of establishing (describing) infinite groups all of whose proper subgroups are finite. As is known, this question was first answered negatively in the class of locally finite p-groups ([4]), and then in the broader class of locally soluble groups ([4]). It was also answered negatively in some other classes of groups ([4,5]). On the other hand, it is not difficult to see that if, in the definition of Schmidt groups, the condition of finiteness of all proper subgroups is weakened to the condition of finiteness of only the nonabelian proper subgroups, infinite nonabelian groups with such a weakened definition already appear among locally finite p-groups. In ([4]) a description is given of locally soluble (and also locally finite) groups of this kind. However, the class of generalized Schmidt groups singled out in this way turned out to be very narrow. The definition of $\overline{IN}$-groups proposed here substantially enlarges this class, as is clear from the results formulated below.

1. Below, $\overline{IN}$-groups are considered under the additional condition of local stepwise character, generalizing both the condition of local solubility and the condition of local finiteness. A group is called locally stepwise here if every finitely generated subgroup different from the identity has a subgroup different from the identity of finite index. Locally stepwise groups are, obviously, locally finite groups, locally soluble (in particular, soluble) groups, and also groups possessing an inclusion-ordered complete

system of subgroups with finite jumps, in particular, groups possessing a normal system with finite factors. In the present article a proposition is formulated on the solvability of locally stepwise $\overline{IH}$-groups, by virtue of which, for $\overline{IH}$-groups, the condition of local stepwise-ness turns out to be equivalent both to the condition that they possess an inclusion-ordered complete system of subgroups with finite jumps (in particular, a normal system with finite factors), and to the condition of their local solvability.

In the propositions formulated below, a description is given mainly of that part of the class of locally stepwise $\overline{IH}$-groups which is formed by the non-metahamiltonian groups contained in it. This part of it is naturally regarded as an increment of the class of infinite locally stepwise metahamiltonian groups, which it obtains when the condition defining it, the invariance of all nonabelian subgroups, is weakened to the condition of invariance only of infinite nonabelian subgroups.

Theorem 1. *Every locally stepwise $\overline{IH}$-group is solvable. The commutator subgroup of a nonperiodic locally stepwise $\overline{IH}$-group is a finite abelian group.*

The second assertion of the theorem means, in particular, that if a torsion-free locally stepwise group has no noninvariant nonabelian subgroups, then it is abelian. From the theorem there follows

Corollary. *A nonperiodic locally stepwise $\overline{IH}$-group is a metahamiltonian group.*

Theorem 2. *A locally stepwise $\overline{IH}$-group with infinite commutator subgroup is not a metahamiltonian group. If a locally stepwise $\overline{IH}$-group is not a metahamiltonian group, then it has a normal divisor of finite index which decomposes into the direct product of a finite number of quasicyclic groups (extremal). A non-metahamiltonian locally stepwise $\overline{IH}$-group with finite commutator subgroup has a quasicyclic subgroup of finite index contained in its center (in other words, it is a central extension of a quasicyclic group by a finite group).*

2. Theorem 3. *If the commutator subgroup $\mathfrak{G}'$ of a locally stepwise $\overline{IH}$-group $\mathfrak{G}$ is nonprimary, then it is not a metahamiltonian group (and therefore is extremal); if, moreover, the commutator subgroup $\mathfrak{G}'$ is finite, then it decomposes into the direct product of a finite Hamiltonian group and a nonabelian Sylow p-subgroup (with $p \neq 2$), which is a central extension of a quasicyclic group by a finite abelian group. The commutator subgroup of a finite metahamiltonian group is primary.*

From Theorems 1 and 3 there follows

Corollary 1. *The commutator subgroup of a nonperiodic locally stepwise $\overline{IH}$-group is a finite primary abelian group.*

Corollary 2. *A periodic (locally stepwise) $\overline{IH}$-group $\mathfrak{G}$ with finite commutator subgroup decomposes either into the product $\mathfrak{G} = \mathfrak{N}\mathfrak{R}$, where $\mathfrak{R}$ is a Sylow p-subgroup invariant in $\mathfrak{G}$, and $\mathfrak{A}$ is an abelian group with orders of elements not divisible by p, or into the direct product $\mathfrak{G} = \mathfrak{R} \times \mathfrak{H}$, where $\mathfrak{R}$ is a nonabelian Sylow p-subgroup of $\mathfrak{G}$, which is a central extension of a quasicyclic group by a finite abelian group, and $\mathfrak{H}$ is a finite Hamiltonian group. A finite metahamiltonian group decomposes into the product of a Sylow p-subgroup invariant in it and an abelian group of order not divisible by p.*

Theorem 4. *The class of locally stepwise INH-groups (see the definition at the beginning of the article) is exhausted by infinite Hamiltonian groups and by such nonabelian non-Hamiltonian groups which are finite extensions of quasicyclic groups by finite abelian and finite Hamiltonian groups.*

3. The present section is devoted to locally stepwise $\overline{IH}$-groups with infinite commutator subgroup. In Theorem 2 the extremality of such groups was noted. Some details of their structure are described in the following theorems.

Theorem 5. *If the commutator subgroup $\mathfrak{G}'$ of a locally stepwise $\overline{IH}$-group $\mathfrak{G}$ is infinite, then it has an invariant Sylow p-subgroup $\mathfrak{P}$ of finite index, which is a finite extension (invariant in $\mathfrak{G}$) of a direct product $\mathfrak{R}$ of finitely many quasicyclic groups, where the subgroup $\mathfrak{P}$ is contained in the commutator subgroup $\mathfrak{G}'$, and the factor group $\mathfrak{G}/\mathfrak{P}$ is a nilpotent group containing no more than one nonabelian Sylow subgroup.*

The author has examples confirming that among $\overline{IH}$-groups satisfying the conditions of this theorem there exist both groups with $\mathfrak{P} = \mathfrak{G}'$ and groups with $\mathfrak{P} \neq \mathfrak{G}'$.

Corollary. *A locally stepwise $\overline{IH}$-group $\mathfrak{G}$ with infinite commutator subgroup has an invariant, extremal Sylow p-subgroup $\mathfrak{P}$, and decomposes as a product $\mathfrak{G} = \mathfrak{P}\mathfrak{S}$, where $\mathfrak{S}$ is a finite nilpotent group of order not divisible by p, having no more than one nonabelian Sylow subgroup.*

Comparing this proposition with Corollary 2 of Theorem 3, we obtain the following proposition.

Every periodic locally stepwise $\overline{IH}$-group has an invariant Sylow subgroup complementable in it, with nilpotent complement containing no more than one nonabelian Sylow subgroup.

Theorem 6. *Let $\mathfrak{P}$ be a maximal complete subgroup of a locally stepwise $\overline{IH}$-group $\mathfrak{G}$ with infinite commutator subgroup $\mathfrak{G}'$, and let $Z(\mathfrak{P})$ be the centralizer of the subgroup $\mathfrak{P}$ in $\mathfrak{G}$. Then the following assertions are true:*

1) *the factor group $\mathfrak{G}/Z(\mathfrak{P})$ is a (finite) cyclic group, and each of its non-identity elements induces in $\mathfrak{P}$ an irreducible automorphism, all automorphisms obtained in this way being distinct;*
2) *if $\mathfrak{P} \neq \mathfrak{G}'$ and the group $\mathfrak{G}$ is not an extension of a quasicyclic group by means of a finite Hamiltonian group, then the factor group $\mathfrak{G}/Z(\mathfrak{P})$ is primary;*
3) *if the subgroup $\mathfrak{P}$ is not a quasicyclic group, then the centralizer $Z(\mathfrak{P})$ is abelian and $\mathfrak{G}/Z(\mathfrak{P})$ is a cyclic group of odd order.*

A nonidentity automorphism of the group $\mathfrak{P}$ is called **irreducible** if there do not exist in it infinite complete subgroups different from it and admissible with respect to this automorphism.

From Theorems 5 and 6 it follows that

Corollary. *Let the commutator subgroup $\mathfrak{G}'$ of a locally stepwise $\overline{IH}$-group $\mathfrak{G}$ be infinite and not coincide with its maximal complete subgroup $\mathfrak{P}$. If the group $\mathfrak{G}/Z(\mathfrak{P})$ is primary and q is the corresponding prime, then every Sylow r-subgroup of the group $\mathfrak{G}$, with $r \neq q$ and $r \neq p$ (p is the prime corresponding to the subgroup $\mathfrak{P}$), is a direct factor of the group $\mathfrak{G}$; such a subgroup is abelian for $r \neq 2$, and for $r = 2$ may be either abelian or Hamiltonian. If the subgroup $\mathfrak{P}$ is not a quasicyclic group, then all such subgroups are abelian.*

Theorems 5 and 6 contain the following assertion.

In order that an infinite nonabelian group $\mathfrak{G}$, not a finite extension of a quasicyclic group, be a locally stepwise IH-group with infinite commutator subgroup, it is necessary that the following conditions be satisfied:

1) the group $\mathfrak{G}$ has an abelian normal divisor $\mathfrak{C}$, defining a finite cyclic factor group $\mathfrak{G}/\mathfrak{C}$ of odd order and being a finite extension of a p-group $\mathfrak{P}$, which decomposes as a direct

a product of quasicyclic groups with a finite number of factors different from the identity,

2) all elements different from the identity of the factor group $\mathfrak{G}/\mathfrak{C}$ induce in the subgroup $\mathfrak{P}$ distinct irreducible automorphisms.

Using Theorems 5 and 6, it is not difficult to be convinced of the necessity, for this, also of the condition:

3) if the factor group $\mathfrak{G}/\mathfrak{P}$ is nonabelian, then the group $\mathfrak{G}/\mathfrak{C}$ is primary (let r be the corresponding prime) and there exists a direct decomposition $\mathfrak{G} = \mathfrak{R} \times \mathfrak{S}$, in which $\mathfrak{S}$ is a finite abelian group whose order is not divisible by r, and $\mathfrak{R}$ is a group containing $\mathfrak{P}$ such that the factor group $\mathfrak{R}/\mathfrak{P}$ is a nonabelian r-group with the following properties:

a) the group $\mathfrak{R}/\mathfrak{P}$ decomposes into the product of two abelian subgroups invariant in it, $\mathfrak{A}/\mathfrak{P}$ and $\mathfrak{B}/\mathfrak{P}$, where the subgroup $\mathfrak{A}$ is abelian and

$\mathfrak{B} = \mathfrak{P}\{B\}$, where B is an r-element for which $\mathfrak{C}\{B\} = \mathfrak{G}$,

b) every cyclic subgroup $\mathfrak{D}/\mathfrak{P}$ of the group $\mathfrak{R}/\mathfrak{P}$ not contained in $\mathfrak{A}/\mathfrak{P}$ is invariant in $\mathfrak{R}/\mathfrak{P}$ and determines the abelian factor group

$$(\mathfrak{R}/\mathfrak{P})/(\mathfrak{D}/\mathfrak{P}).$$

It is not difficult to verify that if an infinite nonabelian group $\mathfrak{G}$ satisfies conditions 1)–3), then it is a locally stepwise $\overline{IH}$-group with infinite commutant. Thus, the following is true.

Theorem 7. *In order that an infinite nonabelian group which is not a finite extension of a quasicyclic group be a locally stepwise $\overline{IH}$-group with infinite commutant, it is necessary and sufficient that conditions 1)–3) be fulfilled.*

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR
Kiev

Received
5 III 1970

REFERENCES CITED

1. S. N. Chernikov, DAN, **171**, No. 4, 806 (1966).

2. G. M. Romalis, N. F. Sesekin, Mat. zap. Ural. (Sverdlovsk) Univ., **5**, 3, 101 (1966).

3. N. F. Sesekin, G. M. Romalis, ibid., **6**, 3, 50 (1968).

4. S. N. Chernikov, a) Ukr. matem. zhurn., **19**, No. 6, 111 (1967); b) DAN, **159**, No. 4, 759 (1964); c) Mat. sborn., **6** (48): 2, 199 (1939); d) **7** (49): 1, 35 (1940); e) **52** (94): 1, 647 (1960).

5. M. I. Kargapolov, Sibirsk. matem. zhurn., **4**, No. 1, 232 (1963).

Note: Figure translations are in progress. See original paper for figures.