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MATHEMATICS

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A PRIORI ESTIMATES FOR THREE-LEVEL DIFFERENCE SCHEMES

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1. In this paper a priori estimates and sufficient conditions for stability with respect to the right-hand side are obtained for three-level difference schemes defined as the operator-difference equation

$$B_2 y^{n+2} + B_1 y^{n+1} + B_0 y^n = \varphi^n, \qquad n = 0, 1, \dots, \tag{1}$$

where B_α are linear operators acting in a linear normed space H, $y^n \in H$, $\varphi^n \in H$. It is assumed that the initial values $y^0, y^1 \in H$ are given and that the operator B_2^{-1} exists.

Introducing the grid $\omega_\tau = \{t_n = n\tau,\ n = 0, 1, \dots, \tau > 0\}$, we write equation (1) in canonical form (see ([1]))

$$By_{\overset{\circ}{t}} + \tau^2 R y_{\bar{t}t} + Ay = \varphi^n, \qquad n = 0, 1, \dots, \quad y^0, y^1 \in H, \tag{2}$$

where $y = y^{n+1} = y(t_{n+1})$, $y_{\overset{\circ}{t}} = (y^{n+2} - y^n)/(2\tau)$, $y_{\bar{t}t} = (y^{n+2} - 2y^{n+1} + y^n)/\tau^2$,

$$A = B_2 + B_1 + B_0, \qquad R = 0.5(B_2 + B_0), \qquad B = \tau(B_2 - B_0).$$

The stability of scheme (2) was studied in the works of A. A. Samarskii ([1],[2]) by means of the method of energy inequalities. The estimates obtained in the present paper are based on reducing (2) to an equivalent two-level scheme and on using the method of separating stationary inhomogeneities ([4]). This made it possible to simplify the a priori estimates and to eliminate some restrictions inherent in the method of energy inequalities. In addition, a number of new estimates are obtained.

2. We shall assume that H is a Hilbert space (real or complex) in which the scalar product (y, v) and the norm $\|y\| = \sqrt{(y, y)}$ are defined.

An operator $A : H \to H$ is called nonnegative ($A \geqslant 0$) if $(Ax, x) \geqslant 0$ for all $x \in H$, and positive ($A > 0$) if $(Ax, x) > 0$ for all $0 \neq x \in H$. If $A^* = A > 0$, then one may introduce the space H_A, consisting of elements $y, v, \dots, \in H$, with scalar product $(y, v)_A = (Ay, v)$ and norm $\|y\|_A = \sqrt{(Ay, y)}$.

In addition to the basic space H, we shall consider the space $H^2 = H \times H$, whose elements are vectors $x = \{x^1, x^2\}$, $x^\alpha \in H$; addition and multiplication by a number are defined componentwise, and the scalar product and norm are given as follows:

$$(y, v) = (y^1, v^1) + (y^2, v^2), \qquad \|y\| = \sqrt{(y, y)}, \qquad y = \{y^1, y^2\}, \quad v = \{v^1, v^2\}.$$

An operator $C = (C_{\alpha\beta}) : H^2 \to H^2$ is defined as a matrix with elements $C_{\alpha\beta} : H \to H$, i.e.

$$Cx = \{C_{11}x^1 + C_{12}x^2,\ C_{21}x^1 + C_{22}x^2\}, \qquad C = (C_{\alpha\beta}), \quad x = \{x^1, x^2\}.$$

We represent scheme (2) in the form of a two-level scheme in the space H^2

$$y_{n+1} = Sy_n + \varphi_n, \qquad n = 0, 1, \dots, \qquad y_0 = \{y^0, y^1\} \in H^2, \tag{3}$$

where $y_n = \{y^n, y^{n+1}\}$, $\varphi_n = \{0, B_2^{-1}\varphi^n\}$, $S = (S_{\alpha\beta})$,

$$S_{11} = 0, \qquad S_{12} = E, \qquad S_{21} = -B_2^{-1}B_0, \qquad S_{22} = -B_2^{-1}B_1, \tag{4}$$

$$B_2 = B/(2\tau) + R, \qquad B_1 = A - 2R, \qquad B_0 = -B/(2\tau) + R. \tag{5}$$

Using the methods of [5], one can show that the following is true.

Lemma. Let operators $A, B, R : H \to H$ be given, with A and R self-adjoint operators. Define the operator $S = (S_{\alpha\beta})$ with components (4) and the operator $D = (D_{\alpha\beta})$ with components

$$D_{11} = D_{22} = R, \qquad D_{12} = D_{21} = 0.5A - R. \tag{6}$$

Then, if the operator inequalities

$$A > 0, \qquad 4R - A \geq 0, \qquad \operatorname{Re} B = 0.5(B + B^*) \geq 0, \tag{7}$$

hold, then D is a nonnegative operator in H^2, and for any $x = \{x^1, x^2\} \in H^2$ the estimate

$$\|Sx\|_D \le \|x\|_D, \tag{8}$$

is valid, where

$$\|x\|_D^2 = (Dx, x) = \tfrac{1}{4}\|x^2 + x^1\|_A^2 + \|x^2 - x^1\|_{R-\frac{1}{4}A}^2.$$

In what follows we shall assume that the operators $A = A(t_n)$ and $4R - A = 4R(t_n) - A(t_n)$ satisfy the following Lipschitz conditions with respect to t:

$$(1 - c_1\tau)A(t_{n-1}) \le A(t_n) \le (1 + c_1\tau)A(t_{n-1}), \quad n = 1, 2, \dots,$$

$$4R(t_n) - A(t_n) \le (1 + c_1\tau)(4R(t_{n-1}) - A(t_{n-1})), \tag{9}$$

where $c_1 > 0$ is a constant independent of τ and n.

3. We now show that conditions (7), (9) ensure the stability of scheme (2) with respect to the right-hand side.

Theorem 1. Let, in scheme (2), $A(t_n)$ and $R(t_n)$ be self-adjoint operators satisfying the Lipschitz conditions (9). If, for each n, the stability conditions (7) hold, then for the solution of problem (2) the estimate

$$\|y_{n+1}\|_{D(t_n)} \le \rho_1^n(\|y_0\|_{D(0)} + \|\varphi^0\|_{A^{-1}(0)}) + \|\varphi^n\|_{A^{-1}(t_n)} +$$

$$+ \sum_{n'=1}^{n} \tau\rho_1^{n-n'}[\|\varphi_{\bar t}^{n'}\|_{A^{-1}(t_{n'})} + c_1\|\varphi^{n'-1}\|_{A^{-1}(t_{n'})}], \tag{10}$$

holds, where

$$\rho_1 = \sqrt{1 + c_1\tau}, \qquad \|\varphi\|_{A^{-1}}^2 = (A^{-1}\varphi, \varphi), \qquad \varphi_{\bar t} = (\varphi^n - \varphi^{n-1})/\tau,$$

$$\|y_{n+1}\|_{D(t_n)}^2 = \tfrac{1}{4}\|y^{n+1} + y^n\|_{A(t_n)}^2 + \|y^{n+1} - y^n\|_{R(t_n)-\frac{1}{4}A(t_n)}^2. \tag{11}$$

Proof. Represent the solution of problem (3) in the form of the sum

$$y_n = v_n + w_n, \qquad n = 0, 1, \dots, \tag{12}$$

where the functions w_n and v_n satisfy the equations

$$(E - S(t_n))w_{n+1} = \varphi_n, \qquad n = 0, 1, \dots, \qquad w_0 = w_1, \tag{13}$$

$$v_{n+1} = S(t_n)v_n + S(t_n)(w_{n+1} - w_n), \qquad n = 0, 1, \dots, \qquad v_0 = y_0 - w_1. \quad (14)$$

Using the lemma given in item 2, we verify the validity of the estimate

$$\|v_{n+1}\|_{D(t_n)} \leq \|S(t_n)v_n\|_{D(t_n)} + \|S(t_n)(w_{n+1} - w_n)\|_{D(t_n)} \leq$$

$$\leq \|v_n\|_{D(t_n)} + \|w_{n+1} - w_n\|_{D(t_n)}.$$

Next, taking into account the Lipschitz conditions (9), we obtain

$$\|v_n\|_{D(t_n)} \leq \rho_1 \|v_n\|_{D(t_{n-1})}, \qquad n = 1, 2, \dots, \qquad \rho_1 = \sqrt{(1 + c_1\tau)}.$$

so that

$$\|v_{n+1}\|_{D(t_n)} \leqslant \rho_1 \|v_n\|_{D(t_{n-1})} + \|w_{n+1} - w_n\|_{D(t_n)}.$$

The solution of equation (13) is the vector

$$w_{n+1} = \{A^{-1}(t_n)\varphi^n, A^{-1}(t_n)\varphi^n\}$$

and, consequently,

$$w_{n+1} - w_n = \tau\{(A^{-1}\varphi^n)_{\bar{t}}, (A^{-1}\varphi^n)_{\bar{t}}\}.$$

An elementary calculation shows that

$$\|w_{n+1} - w_n\|_{D(t_n)} = \tau \|(A^{-1}(t_n)\varphi^n)_{\bar{t}}\|_{A(t_n)}.$$

We now use the following estimate, obtained in (4):

$$\|(A^{-1}\varphi^n)_{\bar{t}}\|_{A(t_n)} \leqslant \|\varphi^n_{\bar{t}}\|_{A^{-1}(t_n)} + c_1 \|\varphi^{n-1}\|_{A^{-1}(t_n)}.$$

Thus, for the solution of problem (14) the estimate

$$\|v_{n+1}\|_{D(t_n)} \leqslant \rho_1 \|v_n\|_{D(t_{n-1})} + \tau \left[\|\varphi^n_{\bar{t}}\|_{A^{-1}(t_n)} + c_1 \|\varphi^{n-1}\|_{A^{-1}(t_n)} \right],$$

is valid, whence we obtain

$$\|v_{n+1}\|_{D(t_n)} \leqslant \rho_1^n \|v_0\|_{D(0)} + \sum_{n'=1}^{n} \tau \rho_1^{n-n'} \left[\|\varphi_{\bar{t}}^{n'}\|_{A^{-1}(t_{n'})} + c_1 \|\varphi^{n'-1}\|_{A^{-1}(t_{n'})} \right].$$

Finally, from the inequality

$$\|y_{n+1}\|_{D(t_n)} \leqslant \|w_{n+1}\|_{D(t_n)} + \|v_{n+1}\|_{D(t_n)} = \|\varphi^n\|_{A^{-1}(t_n)} + \|v_{n+1}\|_{D(t_n)},$$

taking into account that

$$\|v_0\|_{D(0)} \leqslant \|y_0\|_{D(0)} + \|\varphi^0\|_{A^{-1}(0)},$$

we obtain (10).

4. In (5), conditions of stability of scheme (2) with respect to the initial data were obtained which are more general than (7), necessary and sufficient. It can be shown that the operating conditions (5) ensure the stability of scheme (2) also with respect to the right-hand side. More precisely, the following holds.

Theorem 2. *Let the operators A, B, R of scheme (2) be self-adjoint and independent of n. Then, if the stability conditions*

$$\widetilde{A} = \frac{\rho^2 - 1}{2\tau\rho} B + \frac{(1-\rho)^2}{\rho} R + A > 0,$$

$$4\widetilde{R} - \widetilde{A} = \frac{\rho^2 - 1}{2\tau\rho} B + \frac{(1+\rho)^2}{\rho} R - A \geqslant 0,$$

$$\frac{1}{\tau}\widetilde{B} = \frac{\rho^2 + 1}{2\tau} B + (\rho^2 - 1) R \geqslant 0,$$

are satisfied with a constant $\rho > 0$, then for the solution of problem (2) the estimate

$$\|y_{n+1}\|_D \leqslant \rho^{n+1} \left(\|y_0\|_D + \|\varphi^0\|_{\widetilde{A}^{-1}} \right) + \rho \|\varphi^n\|_{\widetilde{A}^{-1}} + \sum_{n'=1}^{n} \rho^{\,n+1-n'} \|\varphi^{n'} - \rho\varphi^{n'-1}\|_{\widetilde{A}^{-1}}, \tag{15}$$

is valid, where

$$\|y_n\|_D^2 = \frac{1}{4}\|y^{n+1} + \rho y^n\|_{\widetilde{A}}^2 + \|y^{n+1} - \rho y^n\|_{\widetilde{R} - \frac{1}{4}\widetilde{A}}^2.$$

The proof is only technically more complicated than that of Theorem 1; therefore we shall not give it.

Estimate (15), valid for any $\rho > 0$, is rather cumbersome because the norm $\|y_n\|_D$ depends on ρ. For $\rho \geqslant 1$ one can obtain sufficient stability conditions in the simpler norm (11). To this end, let us write (2) in the form of a two-level scheme

$$\mathcal{B}y_t + \mathcal{A}y = \varphi_n, \tag{16}$$

where $y = y_n = \{0.5(y^{n+1} + y^n),\, y^{n+1} - y^n\}$, $y_t = (y_{n+1} - y_n)/\tau$, $\varphi_n = \{\varphi^n, 0\}$,

$$\mathcal{A} = (\mathcal{A}_{\alpha\beta}), \qquad \mathcal{B} = (\mathcal{B}_{\alpha\beta}),$$

$$\mathcal{A}_{11} = A, \qquad \mathcal{A}_{12} = \mathcal{A}_{21} = 0, \qquad \mathcal{A}_{22} = R - \tfrac{1}{4}A,$$

$$\mathcal{B}_{11} = B + 0.5\tau A, \qquad \mathcal{B}_{12} = -\mathcal{B}_{21} = 2\mathcal{B}_{22} = \tau(R - \tfrac{1}{4}A).$$

Such a representation makes it possible to apply directly certain theorems on the stability of two-layer schemes ([3,4]) to the three-layer scheme (2). Thus, applying Theorem 3 of ([4]) to (16), it is not difficult to show that the following is valid.

Theorem 3. *Let, in scheme (2), the operators A and R be self-adjoint, let the Lipschitz conditions (9) and the operator inequalities be satisfied*

$$A > 0, \qquad R - \tfrac{1}{4}A > 0, \qquad \operatorname{Re} B + \tfrac{1}{2}\tau\,\frac{\rho - 1}{\rho + 1}\,A \geq 0, \qquad \rho \geq 1. \tag{17}$$

Then for the solution of problem (2) the estimate holds

$$\|y_{n+1}\|_{D(t_n)} \leq \tilde{\rho}^{\,n+1}\left(\|y_0\|_{D(0)} + \|\varphi^0\|_{A^{-1}(0)}\right) + \|\varphi^n\|_{A^{-1}(t_n)} +$$

$$+ \sum_{n'=1}^{n} \tau\tilde{\rho}^{\,n+1-n'}\left[\|\varphi_t^{n'}\|_{A^{-1}(t_{n'})} + c_1\|\varphi^{n'-1}\|_{A^{-1}(t_{n'})}\right], \tag{18}$$

where the norm $\|y_{n+1}\|_{D(t_n)}$ is defined by expression (11), $\tilde{\rho} = \rho\sqrt{1 + c_1\tau}$.

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Note: Figure translations are in progress. See original paper for figures.