ON THE STABILITY OF FORCED OSCILLATIONS OF A FLUID

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Abstract

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ON THE STABILITY OF FORCED OSCILLATIONS OF A FLUID

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This note gives a justification of the linearization method in the problem of stability of periodic motions of a viscous incompressible fluid. At the same time, a new method of proof is obtained for the case of stationary flows ([1]).

1. Problem with initial data. Let the Navier–Stokes equations in a bounded three-dimensional domain Ω with boundary S of class C^2, under a given external body force and boundary velocity that are T-periodic in time, have a T-periodic-in-time solution* with velocity vector $v_0(x,t)$. Seeking an arbitrary flow v in the form $v = v_0 + u$, we arrive at the nonlinear equation of perturbations

$$\frac{du}{dt} + \nu A_0 u + B(t)u = Ku, \tag{1}$$

where the following notation is used:

$$A_0 u = -\Pi\Delta u; \qquad Ku = -K_0(u,u); \quad K_0(u,v) = \Pi(u,\nabla)v,$$

$$K_0^0(u,v) = K_0(u,v) + K_0(v,u); \qquad B(t)u = K_0^0(u,v_0).$$

Here Π is the operator of orthogonal projection of the vector space L_2 onto the space $H \subset L_2$, obtained by closing the set of finite smooth solenoidal vectors in Ω.

We shall regard the operators A_0, B, K as defined on the set D_{A_0} of solenoidal vectors of class $W_2^{(2)}$ that vanish on the boundary S. The operator A_0 is self-adjoint and positive definite in H. We denote its energy space by H_1:

$$(u,v)_{H_1} = (A^{1/2}u, A^{1/2}v)_H. \tag{2}$$

Introduce the set M_T of vector-functions $u(t)$ of time $t \in [0,T]$ with values in H such that $u(t) \in D_{A_0}$ for all $t \in [0,T]$ and the vector-functions $u(t), A_0u(t)$

have strong derivatives of all orders with respect to t in H. Define the Hilbert space H_2^T as the closure of the set M_T in the metric generated by the scalar product

$$(u,v)_{H_2^T} = \int_0^T \left[(du/dt, dv/dt)_H + (A_0u, A_0v)_H\right] dt + (u(T), v(T))_{H_1}. \qquad (3)$$

Lemma 1. The vector-function $u \in H_2^T$ is strongly continuous in H_1 for $t \in [0,T]$. Moreover, the estimates

$$\|u(t)\|_{H_1} \le c\|u\|_{H_2^T}; \qquad 0 \le t \le T; \qquad (4)$$

$$\int_0^T \|u(t)\|_{L_p}^{4p/(p-6)}\, dt \le c\|u\|_{H_2^T}^{4p/(p-6)}; \qquad \int_0^T \|D_{xu}(t)\|_{L_q}^{4q/3(q-2)}\, dt \le c\|u\|_{H_2^T}^{4q/3(q-2)}, \qquad (5)$$

hold, where c depends only on the domain Ω, $6 \le p < \infty$, $2 < q \le 6$.

* The existence theorem for a periodic motion is formulated in ([2]).

These inequalities are easily derived with the aid of S. L. Sobolev' s embedding theorems.

Consider the Cauchy problem for equation (1) with the initial condition

$$u(0) = a, \qquad a \in H_1. \qquad (6)$$

By its solution on the time interval $[0,T]$ we shall mean a vector function $u \in H_2^T$ satisfying equation (1) for almost all $t \in [0,T]$ and the initial condition (6)—in the sense that

$$\|u(t) - a\|_{H_1} \to 0 \qquad (t \to 0). \qquad (7)$$

Similarly, we define the solution of the Cauchy problem for the linearized equation

$$du/dt + A_0u + B(t)u = 0. \qquad (8)$$

Let $\widetilde{H}_2^T$ denote the closure of the set of smooth solenoidal vectors $v(x,t)$ ($x \in \Omega$, $t \in [0,T]$) with respect to the norm

$$\|v\|^2_{\widetilde{H}_2^T} = \int_0^T \left[\|\partial v/\partial t\|^2_{L_2(\Omega)} + \|v\|^2_{W_2^{(2)}(\Omega)} \right] dt. \tag{9}$$

Lemma 2. *Let $v_0 \in \widetilde{H}_2^T$. Then, for any $a \in H_1$, the problems (1, 6) and (8, 6) are uniquely solvable.*

This lemma is not difficult to prove by the method of successive approximations, regarding the terms $B(t)u$ and Ku as perturbations. The necessary estimates follow from Lemma 1 and the identity

$$\|u(t)\|^2_{H_1} + \int_0^t \left(\left\| \frac{du}{dt} \right\|^2_H + \|A_0 u\|^2_H \right) d\tau = \|a\|^2_{H_1} + \int_0^T \|f(\tau)\|^2_H \, d\tau;$$

$$f = \frac{du}{dt} + A_0 u; \qquad a = u(0). \tag{10}$$

Introduce the operators N_t, U_t of displacement along the trajectories of equations (1) and (8), putting, for any $a \in H_1$,

$$N_{ta} = u(t); \qquad U_{ta} = \tilde{u}(t), \tag{11}$$

where $u, \tilde{u}$ are the solutions of the Cauchy problem (1, 6) and the problem (8, 6).

Lemma 3. *The operator $N_t : H_1 \to H_1$ $(0 \le t \le T)$ is defined in a neighborhood of zero in the space H_1, is completely continuous and continuously differentiable. Its Fréchet differential at the point 0 is U_t. The operator U_t is also completely continuous in H_1.*

2. Stability condition. We shall say that the solution v_0 is Lyapunov stable in the space H_1 if problem (1, 6), for every a from some neighborhood of zero in the space H_1, has a solution on the time interval $[0, \infty)$, and for every $\varepsilon > 0$ one can indicate a $\delta > 0$ such that from the condition $\|a\|_{H_1} < \delta$ it follows that $\|N_{ta}\|_{H_1} < \varepsilon$ for $t \ge 0$. If, in addition, $\|N_{ta}\|_{H_1} \to 0$ $(t \to \infty)$, then we shall say that the flow v_0 is asymptotically stable in H_1.

Seeking a solution of equation (6) in the form $u(t) = e^{\sigma t} w(t)$, where $w(t)$ is a T-periodic vector function, we arrive at the problem

$$\frac{dw}{dt} + \sigma w + A_0 w + B(t) w = 0; \qquad w(t+T) \equiv w(t). \tag{12}$$

The set of complex numbers σ for which problem (12) has a nonzero solution is called the stability spectrum of the flow v_0 and is denoted by $\Sigma(v_0)$.

If $\sigma \in \Sigma(v_0)$, then $\rho = e^{T\sigma}$ is an eigenvalue of the monodromy operator U_T; conversely, if ρ is an eigenvalue of the operator U_T, then

$$\sigma_k = \frac{1}{T} \ln \rho + \frac{2\pi}{T} K_i \in \Sigma(v_0), \qquad k = 0, \mp 1, \ldots$$

From the complete continuity of the operator U_T it follows that it can have no more than a finite number of eigenvalues with modulus greater than 1. Therefore the stability spectrum contains no more than a finite number of eigenvalues σ with positive and distinct real parts.

Theorem 1. *Let the stability spectrum of the periodic flow v_0 lie in the left half-plane*

$$\operatorname{Re} \sigma < -\sigma_0 < 0; \qquad \sigma \in \Sigma(v_0). \tag{13}$$

Then the flow v_0 is asymptotically stable in H_1. Moreover, for every solution of the Cauchy problem (1,6), for sufficiently small $\|a\|_{H_1}$, the estimates

$$\|u(t)\| \le C e^{-\sigma_0 t} \|a\|_{H_1}; \qquad \int_0^t e^{2\sigma_0 \tau} \left(\left\| \frac{du}{dt} \right\|_H^2 + \|A_0 u\|_H^2 \right) d\tau \le C^2 \|a\|_{H_1}^2. \tag{14}$$

hold.

The proof of this theorem is obtained by applying to the operator $N_T = N$ the following lemma.

Lemma 4. *Let N be a continuously differentiable operator mapping a neighborhood of zero of the B-space X into X. Let $N(0) = 0$, $N'(0) = U$, and let the spectrum of the operator U be contained inside the unit disk*

$$|\sigma(U)| < \rho < 1. \tag{15}$$

Then $\|N^n x_0\| \to 0$ $(n \to \infty)$, if $\|x_0\|$ is sufficiently small. Moreover, the estimate

$$\|N^n x_0\| \le C \rho^n \|x_0\| \tag{16}$$

holds.

3. Condition of instability. Theorem 2. *Let the stability spectrum $\Sigma(v_0)$ contain at least one eigenvalue σ_0 with positive real part. Then the flow v_0 is unstable in H_1.*

This theorem is derived from the following lemma.

Lemma 5. *Let N, U be the same as in Lemma 4. Let the spectrum of the operator U be represented as the union of nonintersecting closed sets $\sigma_1(U)$ and $\sigma_2(U)$, where*

$$|\sigma_1(U)| > 1; \qquad |\sigma_2(U)| \leq 1. \tag{17}$$

Then there exists $\varepsilon_0 > 0$ such that, for any $\delta > 0$, one can specify a vector $a \in X$ and a natural number n for which

$$\|a\| < \delta; \qquad \|N^n a\| \geq \varepsilon_0. \tag{18}$$

4. Conditional stability. Theorem 3. *Let $\Sigma(v_0)$ contain no points of the imaginary axis. Then, in a neighborhood of zero of the space H_1, there are defined a finite-dimensional manifold Y_1 and a manifold of finite codimension Y_2, possessing the following properties. 1) If $a \in Y_2$, then $N_t a \to 0$ in H_1 as $t \to +\infty$. 2) If $\|a\|_{H_1}$ is small and $a \notin Y_2$, then there exists $t > 0$ such that $\|N_t a\|_{H_1} \geq \varepsilon_0$; $\varepsilon_0 > 0$ does not depend on a. 3) If $a \in Y_1$, then the Cauchy problem (1,6) has a solution $N_t a$, defined for all $t < 0$, and $\|N_t a\|_{H_1} \to 0$ as $t \to -\infty$. 4) If $a \notin Y_1$ and $\|a\|_{H_1}$ is small, then either $\|N_t a\|_{H_1} \geq 0$ for some $t < 0$, or $N_t a$ is not defined for some $t < t_0 \leq 0$.*

This theorem follows from the following lemma.

Lemma 6. Let N, U be the same as in Lemma 4, and suppose the spectrum of the operator U can be represented as the union of closed sets $\sigma_1(U)$ and $\sigma_2(U)$, with

$$|\sigma_1(U)| > 1; \qquad |\sigma_2(U)| < 1. \tag{19}$$

Then, in some neighborhood $D_r = \{x \in X : \|x\| < r\}$; $r > 0$ of zero in the space X, there are manifolds Y_1 and Y_2, invariant with respect to the operator N, which at zero are tangent respectively to the invariant subspaces X_1, X_2 of the operator U corresponding to the spectral sets $\sigma_1(U)$ and $\sigma_2(U)$. Moreover: 1) $\|N^n x\| \to 0$ if $x \in Y_2$; 2) for every $x \in D_r - Y_2$ there exists a natural number n such that $\|N^n x\| \geq r$; 3) for every $x \in Y_1$ the preimage $N^{-1}x$ is uniquely determined, and $\|N^{-n}x\| \to 0$ as $n \to +\infty$; 4) if $x \in D_r - Y_1$, then either, starting from some n, the element $N^{-n}x$ is not defined, or $\|N^{-n}x\| \geq r$ for some n.

The proof of this lemma is close to that given in ([3]) for the finite-dimensional case.

5. **Examples.** 1) If the velocity v_0 is small: $\|v_0\|_{L_3(\Omega)} < Cv$ (C is an absolute constant), then the flow v_0 is stable. 2) The conditions of Theorems 1 and 2 are preserved under a small perturbation of the basic flow. Therefore, from each example of a stable or unstable stationary flow ([4–11]) one obtains corresponding examples of periodic flows.

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Note: Figure translations are in progress. See original paper for figures.