ERGODIC THEOREMS FOR MULTICHANNEL SERVICE SYSTEMS

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Abstract

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ERGODIC THEOREMS FOR MULTICHANNEL SERVICE SYSTEMS

1. Let, on the underlying probability space $(\Omega, \mathfrak{F}, P)$, a sequence of pairs of random variables $\{\tau_j^e, \tau_j^s;\ j \geqslant \infty\}$ be given. We shall say that an m-channel service system with losses is governed by this sequence if calls enter the system at the times $0, \tau_1^e, \tau_1^e + \tau_2^e, \ldots$, and the servicing of the j-th call (if such servicing takes place) requires time τ_j^s. If an arriving call finds all m channels busy, then it is refused and drops out of consideration. If, however, the number q_n of busy channels at the moment just before the arrival of the n-th call is less than m, then the call is accepted for service by one of the free channels. Thus, for the systems under consideration one always has $q_n \leqslant m$. For simplicity we shall assume that $q_1 = 0$ (see Sec. 4).

Along with the "occupancy" q_n, we shall consider more precise characteristics of the systems—the processes $\{q_n(x);\ n \geqslant 1,\ x \geqslant 0\}$, where the value $q_n(x)$ indicates how many of those calls that were in the system before the moment $t_n = \tau_1^e + \cdots + \tau_{n-1}^e$ of arrival of the n-th call will remain in it after time x (i.e., at the moment $t_n + x$), so that $q_n = q_n(0)$.

Suppose that the governing sequence is strictly stationary. Then, without loss of generality, one may assume that on $(\Omega, \mathfrak{F}, P)$ there is given a "two-sided infinite" stationary sequence

$$\{\tau_j^e, \tau_j^s;\ -\infty < j < \infty\}, \tag{1}$$

by which the system is governed.

2. Let first the number of channels be $m = \infty$. Denote by $I(A)$ the indicator of the event A,

$$\begin{aligned} q^k(x) = I(\tau_k^s > \tau_k^e + x) + I(\tau_{k-1}^s > \tau_{k-1}^e + \tau_k^e + x) + \\ + I(\tau_{k-2}^s > \tau_{k-2}^e + \tau_{k-1}^e + \tau_k^e + x) + \cdots \end{aligned} \tag{2}$$

Theorem 1. *If the sequence (1) is strictly stationary,* $\mathbf{M}\tau^s < \infty$*, and the sequence* $\{\tau_j^e;\ -\infty < j < \infty\}$ *is, moreover, metrically transitive, then the distribution of the processes*

$$\{q_{n,k}(x);\ k \geqslant 0;\ x \geqslant 0\} = \{q_{n+k}(x);\ k \geqslant 0,\ x \geqslant 0\}$$

converges monotonically, as $n \to \infty$*, to the distribution of a proper stationary process in* k*,*

$$\{q^k(x);\ k \geqslant 0,\ x \geqslant 0\}.$$

More precisely: for every n *there exists a process* $\tilde{q}^k(x)$ *with the same distribution as* $q^k(x)$*, and such that* $q_{n+k}(x) \leqslant \tilde{q}^k(x)$*, it increases with* n*, and as* $n \to \infty$

$$\mathbf{P}\left(\bigcup_{k=0}^{\infty} \bigcup_{x \geqslant 0} \{q_{n+k}(x) \neq \tilde{q}^k(x)\}\right) \to 0.$$

The proof of this theorem is not difficult to obtain, using the stationarity of the governing sequence and the representation

$$q_n(x) = I(\tau_1^s > \tau_1^e + ... + \tau_{n-1}^e + x) + I(\tau_2^s > \tau_2^e + ... + \tau_{n-1}^e + x) + ...$$

$$... + I(\tau_{n-1}^s > \tau_{n-1}^e + x).$$

Theorem 2. If $\{\tau_j^e\}$ and $\{\tau_j^s\}$ are two independent sequences of independent identically distributed random variables and $\mathbf{M}\tau^e < \infty$, then the condition $\mathbf{M}\tau^s < \infty$ is necessary and sufficient for the finiteness of $q_n(x)$. The probabilities

$$P_k(x) = \mathbf{P}(q^0(x) = k), \qquad k = 0, 1, ... ,$$

satisfy the equations

$$P_0(x) = \int_0^\infty d\mathbf{P}(\tau^e < t)\,\mathbf{P}(\tau^s < t + x) P_0(t + x),$$

$$P_k(x) = \int_0^\infty d\mathbf{P}(\tau^e < t)\,\mathbf{P}(\tau^e \geq t + x) P_{k-1}(x) +$$

$$+ \int_0^\infty d\mathbf{P}(\tau^e < t)\,\mathbf{P}(\tau^s < t + x) P_k(t + x), \qquad k = 1, 2, ... ,$$

which, in the class of systems of functions of bounded variation possessing the properties $P_0(x) \to 1$, $P_k(x) \to \infty$, for $k \geq 1$, $x \to \infty$, have a unique solution.

Hence it is easy to obtain, in particular, that if $\mathbf{P}(\tau^e \geq x) = e^{-\alpha x}$, $\alpha > 0$, then

$$P_k(x) = \frac{1}{k!} \left(\int_x^\infty \alpha \mathbf{P}(\tau^s \geq t)\, dt \right)^k \exp \left\{ - \int_x^\infty \alpha \mathbf{P}(\tau^s \geq t)\, dt \right\}.$$

The "occupancy" $q(t, x)$ at time t, i.e., the number of calls that are in the system at time $t + x$ and whose service has already lasted more than x, has analogous properties. If $\mathbf{P}(\tau^e \geq x) = e^{-\alpha x}$, then the limiting distributions of $q_n(x)$ and $q(t, x)$, as $n \to \infty$, $t \to \infty$, coincide.

3. The number of channels $m < \infty$. Keeping the notation of the governing sequence and $q_n(x)$, we shall now denote the right-hand side in (2) by $Q_k(x)$. This is, as we have seen, a stationary characteristic of the system governed by the same sequence (1), but with an infinite number of channels. Next denote by A_k the event consisting in the fact that, for some $0 \leq L \leq m-1$, $l_j \geq 1$, $j = 0, L$, such that $\sum_{j=0}^L l_j = m$, the inequalities

$$Q_k(0) \leq m - l_0, \qquad Q_k(\tau^e_{k+1}) \leq m - l_0 - l_1, \dots, \qquad Q_k(\tau^e_{k+1} + \dots + \tau^e_{k+L}) \leq$$

$$\leq m - l_0 - \dots - l_L = 0$$

are satisfied.

Theorem 3. Let the sequence (1) be strictly stationary and metrically transitive. Suppose, in addition, that the condition

$$\mathbf{P}(A_0) > 0. \tag{3}$$

is fulfilled. Then the distribution of the processes $\{q_{n+k}(x);\, k \geq 0,\, x \geq 0\}$, as $n \to \infty$, converges to the distribution of some process stationary in k, $\{q^k(x);\, k \geq 0,\, x \geq 0\}$, such that

$$\mathbf{P}(q^0(0) < m) < 1.$$

Convergence here is understood in the same strong sense as in Theorem 1.

There exist various simple sufficient conditions ensuring the fulfillment of (3).

Consider, for example, the case when $\{\tau^e_j\}$, $\{\tau^s_j\}$ are composed of independent random variables. Then for (3) to hold it is sufficient that

$$\mathbf{P}(\tau^s \le m\tau^e) > 0, \qquad \mathbf{M}\tau^s < \infty. \tag{4}$$

A condition in a certain sense opposite will also be sufficient: for all $x \ge x_0$, $\Delta > 0$,

$$\mathbf{P}(\tau^s \in (x, x+\Delta)) > 0, \qquad \mathbf{M}\tau^s < \infty \tag{5}$$

for some $x_0 > 0$.

We now describe, for independent τ_j^e, τ_j^s, the limiting distribution $q_n(x)$. The trajectory of the nondecreasing step process $q^0(x)$ is, evidently, completely described by the positions of its jumps. Denote by

$$\mu(\underbrace{0, \dots, 0}_{j \text{ times}}, dx_{j+1}, \dots, dx_m)$$

the probability that $q^0(0) = m - j$, and that the $(m-j)$ jumps of the process $q^0(x)$ are contained respectively in the intervals $dx_k = (x_k, x_k + dx_k)$, $k = j+1, \dots, m$ (the use of the symbol dx_k simultaneously to denote a scalar quantity and an interval will not lead to misunderstandings).

Theorem 4. *If (4) or (5) holds, the stationary distribution* $\mu(0, \dots, 0, dx_{j+1}, \dots, dx_m)$, $j = 0, \dots, m$, *satisfies the equation*

$$\begin{aligned}\mu(0, \dots, 0, dx_{j+1}, \dots, dx_m) = \sum_{k=j+1}^{m} \int_0^\infty \mathbf{P}(\tau^e \in dt)\, \mathbf{P}(\tau^s \in t + dx_k)\, M_{1,k} \\ + \int_0^\infty \mathbf{P}(\tau^e \in dt)\, \mathbf{P}(\tau^s \le t)\, M_2 + \int_0^\infty \mathbf{P}(\tau^e \in dt)\, M_3,\end{aligned} \tag{6}$$

where

$$M_{1,k} = \int_{0 \le y_2 \le y_3 \le \dots \le y_{j+1} \le t} \cdots \int \mu(0, dy_2, \dots, dy_{j+1}, dx_{j+1} + t, \dots, dx_{k-1} + t,$$

$$+t,\ dx_{k+1} + t, \dots, dx_m + t).$$

$$M_2 = \int_{0 \le y_2 \le y_3 \le \dots \le y_j \le t} \cdots \int \mu(0, dy_2, \dots, dy_j, dx_{j+1} + t, \dots, dx_m + t),$$

$$M_3 = \int_{0<y_1\le y_2\le\cdots\le y_j\le t} \cdots \int \mu(dy_1, \dots, dy_j, dx_{j+1} + t, \dots, dx_m + t).$$

This equation has a unique solution satisfying the condition

$$\int_{0\le x_1\le\cdots\le x_m} \cdots \int \mu(dx_1, \dots, dx_m) = 1.$$

In these formulas, by the interval dy when $y = 0$ one should understand the point 0. Thus the integrals $M_{1,k}$ and M_2 also include "discrete" values, such as $\mu(0, \dots, 0, dx_{j+1} + t, \dots, dx_m + t)$ (in M_2) and others. They do not enter into M_3.

In the equation for $\mu(0, \dots, 0)$, the terms containing $M_{1,k}$ will be absent on the right-hand side; in the equation for $\mu(dx_1, \dots, dx_m)$ with $x_1 > 0, \dots, x_m > 0$, the term containing M_2 will be absent. The integral M_3 in this equation becomes $\mu(dx_1 + t, \dots, dx_m + t)$.

By direct verification one can establish, for example, that in the case

$$\mathbf{P}(\tau^e \ge x) = e^{-\alpha x}, \qquad M\tau^s = a < \infty$$

$$\mu(0, \dots, 0, dx_{j+1}, \dots, dx_m) = ca^{m-j}\mathbf{P}(\tau^s \ge x_{j+1}) \cdots \mathbf{P}(\tau^s \ge x_m)\, dx_{j+1} \cdots dx_m,$$

$$c = \left[\sum_{k=0}^{m} \frac{(a\alpha)^k}{k!}\right]^{-1},$$

satisfies equation (6) and, consequently, is the stationary distribution $q^k(x)$.

As in the case $m = \infty$, here one can also establish that, for the exponential distribution of τ^e, the limiting distributions $q_n(x)$ and $q(t, x)$ as $n \to \infty$, $t \to \infty$ coincide. From this, in particular, will follow Sevast'yanov's theorem concerning Erlang' s formulas for the limiting, as $t \to \infty$, distribution $q(t, x)$.

When τ_j^e, τ_j^s are independent, one can also estimate the rate of convergence of the distribution $q_n(x)$ to the stationary one. If, for example, $\mathbf{P}(\tau^s \le \tau^e) > 0$ and $Me^{\lambda\tau^s} < \infty$ for some $\lambda > 0$, then this rate of convergence is related to an exponential one.

For $m = 1$, Theorem 3 implies

Theorem 5. *If the sequence (1) is strictly stationary and metrically transitive and*

$$\mathbf{P}(\tau_0^s \le \tau_0^e,\ \tau_{-1}^s \le \tau_{-1}^e + \tau_0^e,\ \tau_{-2}^s \le \tau_{-2}^e + \tau_{-1}^e + \tau_0^e, \dots) > 0,$$

then there exists

$$p(x) = \lim_{n\to\infty} \mathbf{P}(q_n(x) = 1). \tag{7}$$

If the vectors (τ_i^e, τ_i^s) are independent and d is the greatest common divisor of the numbers k for which

$$r_k = \mathbf{P}(\tau_1^s \in (X_{k-1}, X_k)) > 0, \qquad X_k = \sum_{j=1}^{k} \tau_j^e,$$

then, for the existence of (7), it is necessary and sufficient that $d = 1$. If this condition is fulfilled,

$$p(x) = \left(\sum_{k=1}^{\infty} k r_k\right)^{-1} \sum_{j=1}^{\infty} \mathbf{P}(\tau_1^s > X_j + x).$$

4. The theorems on the existence of a stationary limiting distribution for the sequence $\{q_{n+k}(x);\ k \ge 0,\ x \ge 0\}$ as $n \to \infty$, formulated above, admit generalization in the following three directions simultaneously:

1) For arbitrary proper initial conditions (at time 0, q_0 channels are busy and the service times of the "initial" calls are equal to $\rho_1, \dots, \rho_{q_0}$).

2) Calls may arrive in batches. In this case, on $(\Omega, \mathfrak{F}, P)$ one should consider strictly stationary sequences, say, of the form $\{\tau_j^e, \nu_j^e, \tau^s;\ -\infty < j < \infty\}$, where ν_j^e is the number of calls arriving in the j-th batch, $\tau^s = (\tau_{j,1}^s, \dots, \tau_{j,\nu_j^e}^s)$ is the vector of service times of the calls in the j-th batch. In this case the condition $M\tau^s < \infty$ in Theorem 1 must be replaced by the requirement $M[\tau^s] < \infty$, where $[\mathbf{x}]$ denotes the sum of the coordinates of the vector $\mathbf{x}$.

3) For the existence of the limiting distribution $q_{n+k}(x)$, the sequence (1) need not be stationary at all. It is sufficient only that, in a certain sense, the sequences $\{\tau_j^e, \tau_j^s;\ j \ge 0\}$ converge as $n \to \infty$ to a stationary one.

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Note: Figure translations are in progress. See original paper for figures.