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Abstract

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EXTERIOR PROBLEMS FOR THE POLY-HARMONIC EQUATION

(Presented by Academician S. L. Sobolev on 24 III 1970)

In the present paper, for the equation

$$\Delta^m u = 0 \tag{1}$$

and an infinite domain Ω with bounded complement in the n-dimensional Euclidean space E_n, the existence and uniqueness of solutions are proved for a number of boundary-value problems in which the same conditions are prescribed on the boundary as in the first boundary-value problem for equation (1) in the case of a bounded domain (^{1}). The problems considered here differ from one another only in the requirements on the behavior of the solutions at infinity. For $n = 2s + 1,\ s = 1, 2, \ldots$, or $n > 2m$, we shall consider $m + 1$ problems for equation (1), while for $n = 2s$ and $2m \geq n$ we shall consider m problems. We shall call them problems $t,\ t = 1, \ldots, m + 1$, or $t = 1, \ldots, m$.

We shall require of Ω that its boundary S be simple (for the definition of a simple boundary see (^{1}), p. 81). By $\dot{W}_2^{(m)}(\Omega)$ we denote the set of functions from $L_2^{(m)}(\Omega)$ that are equal to zero together with their derivatives up to order $m - 1$ on S (equality of functions here and below is understood in the sense of Sobolev embedding theorems (^{1})). By φ we denote an arbitrary function from $L_2^{(m)}(\Omega)$.

Problem ***t*****.** ($t = 1, \ldots, m + 1$, if $2m < n$; $t = 1, \ldots, n/2$, if $n = 2s$ and $2m \geq n$; $t = 1, \ldots, n/2 + 1/2$, if $n = 2s + 1$ and $2m > n$. Find a solution of equation (1) in Ω, u, such that:

a) $[u - \varphi] \in \dot{W}_2^{(m)}(\Omega)$,

(2)

b) $u = P_{m-t} + w$,

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where P_{m-t} is a polynomial of degree not exceeding $m-t$, and $w = O\left(|x|^{-n+m+t-1}\right)$ as $|x| \to \infty$ for $n = 2s+1$, or $-n+m+t-1 < 0$, and $w = O\left(|x|^{-n+m+t-1} \times \ln|x|\right)$ for $n = 2s$ and $-n+m+t-1 \geq 0$.

Let us prove uniqueness of the solution of problem t. Let $u \in \dot{W}_2^{(m)}(\Omega)$ be a solution of problem t. Denote by Ω_R^* the sphere of radius R with surface S_R, centered at the origin, such that $S_R \subset \Omega$. Applying to u Green' s formula for the polyharmonic operator $\left(\left(^2\right),\right.$ p. 73) and the domain $\Omega_R^* \cap \Omega$, we obtain

$$\|u\|^2_{L^{(m)}2(\Omega_R^* \cap \Omega)} = \int_S F_m(u,u)\,dS + \int_{S_R} F_m(u,u)\,dS_R, \tag{3}$$

where $F_m(u,u)$ is a differential form such that

$$|F_m(u,u)| = |F_m(w,u)| <$$

$$< K \sum_{l=m}^{2m-1} \left[\left(\sum_{j_1=1}^{n} \cdots \sum_{j_l=1}^{n} \frac{\partial^l w}{\partial x_{j_1} \cdots \partial x_{j_l}} \right) \left(\sum_{j_1=1}^{n} \cdots \sum_{j_{2m-1-l}=1}^{n} \frac{\partial^{2m-1-l} u}{\partial x_{j_1} \cdots \partial x_{j_{2m-1-l}}} \right) \right]. \tag{4}$$

From the estimates $\left(\left(^1\right),\right.$ §15) it follows that the integral over S in (3) is equal to zero. Cons-

consider the integral over S_R. From the inequalities

$$\left| \frac{\partial^l w}{\partial x_{j_1} \ldots \partial x_{j_l}} \right| < K|x|^{-n+m+l-1-l} \ln|x|,$$

$$\left| \frac{\partial^{2m-1-l} u}{\partial x_{j_1} \ldots \partial x_{j_{2m-1-l}}} \right| < K|x|^{-m-l+1+l}$$

and (4) it follows that

$$\lim_{R \to \infty} \int_{S_R} F_m(u,w)\,dS_R = 0.$$

Passing in (3) to the limit as $R \to \infty$, we obtain $\|u\|_{L_2^{(m)}(\Omega)} = 0$ and the uniqueness of the solution of the problem.

Denote by $Q_m(\Omega)$, $Q_m(\Omega_R)$, where $\Omega_R = E_n \setminus \overline{\Omega_R^*}$, the sets of functions from $\overset{\circ}{W}{}_2^{(m)}(\Omega)$, $\overset{\circ}{W}{}_2^{(m)}(\Omega_R)$ satisfying equation (1).

Lemma 1. $\dim Q_m(\Omega_R) \leq \dim Q_m(\Omega)$.

Proof. Every solution u of equation (1) of the class $L_2^{(m)}(\Omega_R)$ is expanded in Ω_R into the Fourier series ((^{2}), pp. 104–107; (^{3}), pp. 252–253).

$$u = \sum_{k=0}^{\infty} \sum_{l=1}^{l(k)} \left(\sum_{1 \le i < \alpha(k)} a_{k,l}^i f_{k,l}^i + \sum_{\substack{1 \le i < \beta(k) \\ i < m}} b_{k,l}^i g_{k,l}^i \right); \qquad l(k) = \frac{(n+k-3)!}{(n-2)!\,k!}(n+2k-2), \tag{5}$$

where $a_{k,l}^i$, $b_{k,l}^i$ are constants,

$$\alpha(k) = \frac{m+1-k}{2}, \qquad \beta(k) = \frac{m}{2} + \frac{k}{2} + \frac{n}{4},$$

and $f_{k,l}^i$, $g_{k,l}^i$ are expressed in terms of spherical functions of order k, number l, as follows:

$$f_{k,l}^i = Y_{k,l}|x|^{k+2i-2}; \qquad g_{k,l}^i = Y_{k,l}|x|^{-k-n+2i} \quad \text{for } n = 2s+1 \text{ or } i < k + \frac{n}{2};$$

$$g_{k,l}^i = Y_{k,l}|x|^{-k-n+2i} \ln|x| \quad \text{for } n = 2s \text{ and } i \ge k + \frac{n}{2}.$$

Functions of the class $Q_m(\Omega_R)$ are equal to zero on S_R, together with derivatives up to order $m-1$. Setting $|x| = R$ in (5), equating $\partial^l u / \partial |x|^l$, $l = 0, \dots, m-1$, to zero on S_R, and taking into account the linear independence of the $Y_{k,l}$ on the unit sphere, we find that all $u \in Q_m(\Omega_R)$ are linear combinations of $f_{k,l}^i$, $g_{k,l}^i$, $k < m$, and consequently are polyharmonically continued to functions from $L_2^{(m)}(\Omega)$.

Define the **problem** $*$. Let $\varphi \in L_2^{(m)}(\Omega)$. It is required to find a function having minimal norm in $L_2^{(m)}(\Omega)$ among all u satisfying (2). A solution of this problem exists, is unique, and satisfies equation (1) in Ω ((^{1}), (^{2}), p. 99).

Let $f_1, \dots, f_r \in Q_m(\Omega_R)$ and be linearly independent. We shall regard them as polyharmonically continued into Ω. Denote by p_i the solutions of problem $*$ in Ω with conditions on S: $[p_i - f_i] \in \overset{\circ}{W}{}_2^{(m)}(\Omega)$. Put $f_i - p_i = g_i \in Q_m(\Omega)$. It can be established that the g_i are linearly independent in Ω_R, and hence in Ω, and the validity of Lemma 1.

Let $r(t)$ be the number of functions $f_{k,l}^i$, $g_{k,l}^i$ from (5) which are not, for any function φ, solutions of problem t. Denote such functions by $\psi_1^t, \dots, \psi_{r(t)}^t$. The proof of the existence of a solution of problem t reduces to the construction of $u \in L_2^{(m)}(\Omega)$ satisfying (1) and, in its expansion (5), not containing the terms $\psi_1^t, \dots, \psi_{r(t)}^t$. We verify directly that for all t, $r(t) = \dim Q(\Omega_R) = r$. From

Lemma 1 follows the existence of r linearly independent functions $g_i \in Q_m(\Omega)$. Let v be the solution of the prob-

with the conditions on S: $[v - \varphi] \in \dot{W}_2^{(m)}(\Omega)$. Represent v and g_i in the form

$$v = v^0 + \sum_{j=1}^{r} c_j \psi_j^t, \qquad g_i = g_i^0 + \sum_{j=1}^{r} d_{i,j} \psi_j^t,$$

where $c_j, d_{i,j}$ are constants, while v^0, g_i^0 in their expansion in Ω_R of the form (5) do not contain ψ_j^t. We form the system of equations

$$\sum_{i=1}^{r} \beta_i d_{i,j} = c_j, \qquad j = 1, \dots, r. \tag{6}$$

Suppose that the system (6) has no solution. Then the corresponding homogeneous system has a nontrivial solution $\beta_1 = \beta_1^0, \dots, \beta_n = \beta_n^0$, and

$$u = \sum_{i=1}^{r} \beta_i^0 g_i$$

will be a nonzero solution of the problem t, belonging to $\dot{W}_2^{(m)}(\Omega)$, which is impossible by virtue of the uniqueness of this problem. Consequently, the system (6) has a solution $\beta_1 = \beta_1^0, \dots, \beta_n = \beta_n^0$, and the function

$$u = v - \sum_{i=1}^{r} \beta_i^0 g_i$$

is a solution of the problem t.

Remark. Of special interest is the problem t for $t = m + 1$ when $2m < n$, $t = n/2$ when $n = 2s$ and $2m \geq n$, $t = n/2 + 1/2$ when $n = 2s + 1$ and $2m > n$. From (5) it is seen that the indicated problem may be regarded as the problem of finding a solution u satisfying (2) and the condition $u = O(|x|^{m-[(n+1)/2]})$. By analogy with the Laplace equation we shall call it the Dirichlet problem. Its formulation and solution for the case of the exterior of a sphere (see (²), pp. 99, 104-107; (³), pp. 252-253).

We shall assume $2m > n$. Let $\eta(t)$ be the number of all possible functions $Y_{k,l}|x|^{-k+2i-n}$, $i = 1, \dots, m$, such that $m - n/2 < -k + 2i - n \leq m + t - n$ when $n = 2s$, and, when $n = 2s + 1$, such that $m - n/2 < -k + 2i - n < m + t - n - 1$. We denote these functions themselves by $\rho_1^t \dots \rho_{\eta(t)}^t$.

Problem t $\left(t = \frac{n}{2} + 1, \dots, m \text{ when } n = 2s, \quad t = \frac{n}{2} + \frac{1}{2} + 1, \dots, m + 1 \text{ when } n = 2s + 1\right)$. Find a solution of equation (1) in Ω, taking on S, together with its derivatives up to order $m-1$, prescribed values equal there to the values of a function $\varphi \in L_2^{(m)}(\Omega)$ and of its corresponding derivatives, and having the form

$$u = w + \sum_{i=1}^{\eta(t)} \alpha_i \rho_i^t,$$

where α_i are constants, $w \in L_2^{(m)}(\Omega)$, and in its expansion of the form (5) contains no terms $Y_{k,l}|x|^{k+2i-2} \ln|x|$ such that $k+2i-2 \geq m-t$ when $n = 2s$, and no terms $Y_{k,l}|x|^{k+2i-2}$ such that $k+2i-2 \geq m+1-t$.

The proof of existence and uniqueness of the solution of these problems is, on the whole, analogous to the proofs for the problems of the preceding group. Uniqueness is established as follows. Let u be a solution of the problem t with zero boundary conditions. We pass to polar coordinates. We obtain $u = u(|x|, \theta)$, where $\theta \in E_n$, $|\theta| = 1$. The Kelvin transform of u, $K[u] = |x|^{2m-n} u(|x|^{-1}, \theta)$, is a polyharmonic function in the domain Ω^{-1} obtained by inversion from Ω. Let Ω_h^* be a sphere of small radius h with center at the origin. We apply to $K[u]$ Green' s formula for the polyharmonic operator and the domain $\Omega^{-1} \setminus \Omega_h^*$, and, passing to the limit as $h \to 0$, we find that $K(u)$ and u are equal to zero. To prove exist—

of the solution of problem t, we find the solution of the Dirichlet problem v with the same boundary conditions, after which one seeks a function g^t, equal to zero together with its derivatives up to order $m-1$ on S, such that $u = v + g^t$ solves problem t. We seek g^t in the form of a linear combination of the functions $g_1^t, \dots, g_n^t(t)$, having the form $g_i^t = \rho_i^t - p_i^t$, where p_i^t are solutions of the Dirichlet problem that assume on S, together with their derivatives up to order $m-1$, the same values as ρ_i^t. As before, the existence of a solution follows from the solvability of a certain system of algebraic equations; moreover, the fact that the determinant of this system is not equal to zero follows from the uniqueness of the solution of problem t.

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Note: Figure translations are in progress. See original paper for figures.

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