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Abstract

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MATHEMATICS

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**INVESTIGATION OF THE BIFURCATION OF
SMALL EIGENFUNCTIONS IN THE CASE OF
MULTIDIMENSIONAL DEGENERACY**

(Presented by Academician A. Yu. Ishlinskii, May 22, 1970)

In the present paper the possibility of applying, in the general case, a projection transition to a finite-dimensional equation in the problem of bifurcation points is substantiated.

1. We consider a nonlinear operator T acting in a real Banach space E . It is assumed that the operator T is defined in a neighborhood of the zero 0 of the space E , that $T(0) = 0$, and

$$T = A + B + \Omega, \tag{1}$$

where A is a linear operator, B is a homogeneous Fréchet differentiable operator of order k ($k \geq 2$), whose derivative $B'(x)$ satisfies the Lipschitz condition

$$\|B'(x) - B'(y)\| \leq c\rho^{k-2}\|x - y\| \quad (\|x\|, \|y\| \leq \rho), \tag{2}$$

and Ω satisfies the condition

$$\lim_{\|x\| \rightarrow \infty} \|x\|^{-k} \cdot \|\Omega x\| = 0.$$

If, in addition, the condition

$$\|\Omega x - \Omega y\| \leq O(\rho^{k-1})\|x - y\|, \quad (\|x\|, \|y\| \leq \rho),$$

is fulfilled, then we shall call the operator T smooth.

As is known, bifurcation points of the equation

$$Tx = \lambda x \tag{3}$$

can only be points of the spectrum of the linear operator A . Let λ_0 be an isolated nonzero point of the spectrum of the operator A . We shall assume that λ_0 is a Fredholm point of the spectrum in the sense of M. G. Krein ⁽¹⁾. This means that the root subspace E_0 of the operator A corresponding to the eigenvalue λ_0 is finite-dimensional, that the operator $A - \lambda_0 I$ is normally solvable, and that the Noether index of the operator $A - \lambda_0 I$ is equal to zero. All these properties are satisfied if A is completely continuous. We shall assume that the root subspace E_0 consists only of eigenvectors. Then ⁽²⁾ defines a unique operator P of linear projection onto E_0 , commuting with A . The operator P is defined by the simple formula

$$Px = f_1(x)e_1 + \dots + f_n(x)e_n \quad (x \in E), \quad (4)$$

where e_i form a basis in E_0 , and the functionals f_i form a basis in the subspace of zeros of the operator $A^* - \lambda_0 I$, these bases being related by the condition $f_i(e_j) = \delta_{ij}$ (see, for example, ^(3,4)).

The principal role in our constructions is played by the nonlinear operator $B_0 = PB$. This operator is considered only in the n -dimensional space E_0 . If B is defined by the k -th differential of the operator T , then, for any basis in E_0 , the values of the operator B_0 have as their components homogeneous polynomials of order k in the coordinates of the point x . We note,

that in all the statements given below it could be assumed that inequality (2) is satisfied not for the operator B , but for the operator B_0 in the space E_0 .

The ray ty_0 ($t > 0$, $\|y_0\| = 1$, $y_0 \in E_0$) will be called a simple eigenray of the operator B_0 if $B_0 y_0 = \gamma y_0$, $\gamma \neq 0$, and γ is not an eigenvalue of the linear operator $B'_0(y_0)$. The latter condition means that the determinant of the matrix $B'_0(y_0) - \gamma I$ is different from zero.

Theorem 1. *Suppose that the operator B_0 has at least one simple eigenray, and suppose that the operator T is smooth.*

Then λ_0 is a bifurcation point of equation (3).

Theorem 2. *Suppose that the operator B_0 has at least one simple eigenray, and suppose that the operator T is smooth.*

Then λ_0 is a bifurcation point of equation (3).

2. We shall say that the set \mathfrak{R} of nonzero solutions of equation (3) is a branch of eigenvectors of the operator T , directed along the vector y_0 , corresponding to the bifurcation point λ_0 , if \mathfrak{R} contains points of arbitrarily small norm, if

$$\lim_{\|x\| \rightarrow 0} \left\| \frac{x}{\|x\|} - y_0 \right\| = 0, \quad (x \in \mathfrak{R}),$$

if, for $x \in \mathfrak{R}$, the difference $\lambda(x) - \lambda_0$ preserves a constant sign and

$$\lim_{\|x\| \rightarrow 0} \lambda(x) = \lambda_0$$

and, finally, if the values $\lambda(x)$ ($x \in \mathfrak{R}$) fill some interval adjoining the point λ_0 . In this definition, $\lambda(x)$ denotes the value of the parameter λ for which the point $x \in \mathfrak{R}$ is a solution of equation (3).

A directed branch \mathfrak{R} is called a proper directed branch if \mathfrak{R} is the set of values of a single-valued continuous function $x(\lambda)$.

By $K(y_0; \varepsilon)$ we denote the cone in the space E consisting of those elements x for which, for some $t \geq 0$, the inequality $\|x - ty_0\| \leq t\varepsilon$ holds. The set of nonzero solutions of equation (3) lying in $K(y_0; \varepsilon)$ and having norm not greater than ρ will be denoted by $\mathfrak{R}(y_0, \varepsilon, \rho)$.

Theorem 3. *Suppose the conditions of Theorem 1 are satisfied. Suppose that ε and ρ are sufficiently small.*

Then $\mathfrak{R}(y_0, \varepsilon, \rho)$ is a branch of eigenvectors of the operator T , directed along the vector y_j , corresponding to the bifurcation point λ_0 .

Theorem 4. *Suppose the conditions of Theorem 2 are satisfied. Suppose that ε and ρ are sufficiently small.*

Then $\mathfrak{R}(y_0, \varepsilon, \rho)$ is a proper branch of eigenvectors of the operator T , directed along the vector y_0 , corresponding to the bifurcation point λ_0 .

Under the conditions of Theorems 3 and 4, the sign of the difference $\lambda(x) - \lambda_0$ ($x \in \mathfrak{R}(y_0, \varepsilon, \rho)$) coincides with the sign of γ . The same can also be said as follows: the points $x \in \mathfrak{R}(y_0, \varepsilon, \rho)$ are solutions $x = x(\lambda)$ of equation (3) for values of λ sufficiently close to λ_0 such that $\gamma(\lambda - \lambda_0) > 0$. The function $x(\lambda)$ is single-valued only under the conditions of Theorem 4.

Theorem 5. *Suppose the conditions of Theorem 3 or of Theorem 4 are satisfied. Then the solutions $x(\lambda) \in \mathfrak{R}(y_0, \varepsilon, \rho)$ have the asymptotic representation*

$$x(\lambda) = \left(\frac{\lambda - \lambda_0}{\gamma} \right)^{1/(k-1)} y_0 + h(\lambda), \quad (5)$$

where $h(\lambda) = O(|\lambda - \lambda_0|^{1/(k-1)})$ and

$$\|h(\lambda) - Ph(\lambda)\| \leq c|\lambda - \lambda_0|^{k/(k-1)}. \quad (6)$$

3. We shall say that λ_0 is a point of simple bifurcation if there exist positive numbers ε_0 and ρ_0 such that the nonzero solutions of equation (3) for $|\lambda - \lambda_0| < \varepsilon_0$, lying in the ball $\|x\| < \rho_0$, form a finite number of directed branches of eigenvectors. If all these directed branches are regular, then λ_0 is a point of regular bifurcation.

We shall call the operator B_0 regular if each of its eigenvectors $y_0 \in E_0$ ($By_0 = \gamma y_0$) determines a simple eigenray ty_0 ($t > 0$).

Let $Tx_n = \lambda_n x_n$, $\lambda_n \rightarrow \lambda_0$ as $n \rightarrow \infty$. Then the sequence $\|x\|^{-1}x_n$ is compact and its limit points can only be eigenvectors of the operator B_0 . Hence, from Theorems 3 and 4 the following assertions follow.

Theorem 6. Let the operator B_0 be regular, and let the operator T be completely continuous.

Then λ_0 is a point of simple bifurcation of equation (3).

Theorem 7. Let B_0 be regular, and let T be smooth.

Then λ_0 is a point of regular bifurcation of equation (3).

4. If E_0 is two-dimensional, the operator B_0 has the form

$$B_0(\xi e_1 + \eta e_2) = \varphi(\xi, \eta)e_1 + \psi(\xi, \eta)e_2, \quad (7)$$

where e_1 and e_2 are a basis in E_0 , and $\varphi(\xi, \eta)$ and $\psi(\xi, \eta)$ are homogeneous functions (the order of homogeneity is equal to k). Suppose that the vector e_2 is not an eigenvector of the operator B_0 . Then the eigenvectors may be sought in the form $y_0 = e_1 + \eta_0 e_2$, and η_0 is determined from the equation

$$\eta\varphi(1, \eta) - \psi(1, \eta) = 0. \quad (8)$$

In ordinary cases (8) is an algebraic equation of order $k + 1$. A root η_0 of equation (8) is called simple if

$$\varphi(1, \eta_0) + \eta_0 \varphi'_\eta(1, \eta_0) - \psi'_\eta(1, \eta_0) \neq 0.$$

Theorem 8. The vectors $te_1 + t\eta_0 e_2$ ($t > 0$) and $-te_1 - t\eta_0 e_2$ ($t > 0$) form simple eigenrays of the operator (7) if and only if η_0 is a simple root of equation (8).

An important characteristic of the operator (7) is the Poincaré index γ_0 of its singular point. If the functions $\varphi(\xi, \eta)$ and $\psi(\xi, \eta)$ are homogeneous polynomials of order k , then, as is known, in the case of even k , γ_0 can take the values $0, \pm 2, \dots, \pm k$, and for odd k the values $\pm 1, \pm 3, \dots, \pm k$. In particular, for $k = 2$ the number γ_0 may take one of three values: 0, 2, or -2. The computation of γ_0 is carried out ⁽⁵⁾ with the aid of the generalized Sturm sequence $T_0(\eta), T_1(\eta), \dots, T_l(\eta)$, where $T_0(\eta) = \varphi(1, \eta)$, $T_1(\eta) = \psi(1, \eta)$, $T_{i-1}(\eta) = \varepsilon_i(\eta)T_i(\eta) - T_{i+1}(\eta)$ ($i = 1, \dots, l$), and $T_{l-1}(\eta) = \varepsilon_l(\eta)T_l(\eta)$. If the polynomial $T_l(\eta)$ has no real roots, then the operator B_0 is nondegenerate and $\gamma_0 = s(+\infty) - s(-\infty)$, where $s(-\infty)$ denotes the number of sign changes in the sequence of values of the generalized Sturm sequence for negative values of η large

in absolute value, and $s(+\infty)$ is the number of sign changes for large positive η . From Theorem 8 it follows that

Theorem 9. Let E_0 be two-dimensional and let $\gamma_0 = -k$, where k is the order of homogeneity of the operator B_0 .

Then the operator B_0 is regular and has exactly $2k + 2$ simple eigenrays.

5. The theorems of this paper are applicable to the investigation, for example, of nonlinear integral equations. Here we present one unexpected assertion which we encountered in the study of the integral operator

$$Tx(t) = \int_a^b G(t, s) f[s, x(s)] ds. \quad (9)$$

We shall consider this operator in the space C of functions continuous on $[a, b]$. For simplicity we shall assume that the kernel $G(t, s)$ is continuous, symmetric, and positive definite. The function $f(s, u)$ ($a \leq s \leq b$; $|u| \leq \rho$) will be assumed continuous in the aggregate of its variables and of the form

$$f(s, u) = u + g(s)u^2 + o(u^2).$$

Then T can be represented in the form (1), where

$$Ax(t) = \int_a^b G(t, s)x(s) ds, \quad Bx(t) = \int_a^b G(t, s)g(s)x^2(s) ds.$$

If λ_0 is an eigenvalue of the operator A of multiplicity two, to which there correspond two orthogonal and normalized eigenfunctions $e_1(t)$ and $e_2(t)$, then the operator B_0 is determined by the formula

$$\begin{aligned} B_0(\xi e_1 + \eta e_2) = & e_1(t) \int_a^b g(s)[\xi e_1(s) + \eta e_2(s)]^2 e_1(s) ds + \\ & + e_2(t) \int_a^b g(s)[\xi e_1(s) + \eta e_2(s)]^2 e_2(s) ds. \end{aligned} \quad (10)$$

Theorem 10. *The Poincaré index γ_0 , defined by the operator (10), when this operator is nondegenerate, is equal either to zero or to -2 .*

6. Simple proofs of the main theorems of the article were obtained by us using some ideas of A. E. Gelman and V. A. Trenogin, developed by them in the theory of branching of solutions; an extensive bibliography is contained in the handbook (6). It seems to us that the rules set forth above are convenient for applications, since they do not require the construction and investigation of branching equations (see (6)).

The results of this work substantially develop the theorems of our article (7): we have succeeded in freeing ourselves from all three principal restrictions of the work (7); in it it was assumed that $n = 2$, $k = 2$, and that for all points μ of the spectrum of the operator A not coinciding with λ_0 , the inequality $|\mu| < \lambda_0$ holds.

The present work was carried out in connection with the remarks and wishes of G. E. Shilov concerning the work (7); I am sincerely grateful to him.

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