SPACES DEFINED BY THE VARIATION OF LOCAL APPROXIMATIONS

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Abstract

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SPACES DEFINED BY THE VARIATION OF LOCAL APPROXIMATIONS

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1. A family of spaces defined by the variation of local approximations is obtained by means of a very simple construction. Nevertheless, among the members of this family there are a number of spaces that are constantly used in analysis (Lipschitz spaces, functions of bounded variation, Sobolev spaces, etc.). Thus a somewhat unexpected connection between these spaces is established; in particular, from the theorems given below there follows a number of new properties of these spaces. We note that two representatives of the family of spaces under consideration were studied earlier by F. Riesz (Riesz' s lemma, see ([1]), p. 85) and by F. John–L. Nirenberg ([2]).

2. We shall need several definitions. Below Q_0 denotes a fixed n-cube, and $Q \subset Q_0$ a cube parallel to it. Let X be a Banach space whose elements are functions on the cube $I(Q)$, and suppose that the norm in X has the following property: if $|I_1(Q)| \leq |I_2(Q)|$, $Q \subset Q_0$, and $I_2 \in X$, then $I_1 \in X$ and $\|I_1\| \leq \|I_2\|$. Let P be the projector $L_p(Q_0) \to P_{k-1}$, where P_{k-1} is the space of polynomials of degree $k-1$ (consisting only of zero when $k=0$), and let P_Q be the "transplant" of P to $L_p(Q)$ by means of a homothety carrying Q_0 onto Q. Finally, let

$$I_f^k(Q) = \left\{ \frac{1}{m(Q)} \int_Q |f - Pf|^p dx \right\}^{1/p}. \tag{1}$$

Definition. The space $\mathcal{L}_p^k(X)$ is the linear set of functions $f \in L_p(Q_0)$ for which $I_f^k(Q) \in X$. As the norm of f in this space we take the quantity

$$\|f\|_{L_p(Q_0)} + \|I_f^k\|_X.$$

3. We choose as X the space of functions $I(Q)$ having bounded variation in the following sense:

$$V(I;\Omega) = \sup\left\{\sum m(Q_s)\left|\frac{I(Q_s)}{\varphi(a_s)}\right|^q\right\}^{1/q} < +\infty \tag{2}$$

for every open $\Omega \subset Q_0$. Here the supremum is taken over all families $\{Q_s\}$ of pairwise nonoverlapping cubes in Ω; a_s is the side length of Q_s, and φ is a majorant, i.e. a positive nondecreasing function on $(0, +\infty)$ satisfying the condition

$$\sup\frac{\varphi(2\tau)}{\varphi(\tau)} < +\infty.$$

We denote this space by V_q^φ; its subspace consisting of those I for which (2) is absolutely continuous will be denoted by v_q^φ.

For the formulation of the first theorem, set

$$m_Q(f;\tau) = \operatorname{mes}\{x \in Q \mid |(f - P_Q f)(x)| > \tau\} \tag{3}$$

and by $f_Q^*(\tau)$, $0 < \tau \le m(Q)$, the right-continuous function inverse to (3).

Theorem 1. If $f \in \mathcal{L}_p^k(V_q^\varphi)$, $1 \le p, q \le \infty$, then for every $Q \subset Q_0$ with side a

$$f_Q^*(\tau^n) \le cV(I_f^k;Q)\int_\tau^a \frac{\varphi(\tau)}{\tau^{1+n/q}}\,d\tau. \tag{4}$$

From Theorem 1, for $k = 1$ and $\varphi = 1$, we obtain the known results of John–Nirenberg ([2]), and for $k = 1$ and $q = \infty$, the result of S. Spanne ([3]).

Corollary 1. If, for some $r \ge q$, the function

$$\bar\varphi(\tau) = \tau^\lambda\int_0^\tau \frac{\varphi(u)}{u^{1+a}}\,du, \qquad a = n(q^{-1} - r^{-1}),$$

is defined, then there is a continuous embedding

$$\mathcal{L}_p^k(V_q^\varphi) \subset \mathcal{L}_r^k(V_q^{\bar\varphi}).$$

In particular, if $p \le r$ and φ is a quasi-degree majorant, i.e. $\bar\varphi \le \varphi$, then $\mathcal{L}_p^k(V_q^\varphi)$ is isomorphic to $\mathcal{L}_r^k(V_q^\varphi)$.

Corollary 2. All spaces $\mathcal{L}_p^k(V_q^\varphi)$, $1 \le p < q$, are isomorphic.

For the formulation of the second theorem, assume that $\varphi(\tau) = \tau^s\psi(\tau)$, where $s \ge 0$ is an integer smaller than k, and ψ is a modulus of continuity. Put

$$\bar{\psi}(\tau) = \int_0^\tau \frac{\psi(u)}{u}\,du + \tau \int_\tau^1 \frac{\psi(u)}{u^2}\,du, \tag{5}$$

where, for $s = k-1$, the second term is omitted.

Theorem 2. If $f \in \mathcal{L}_p^k(V_q^\varphi)$, $\varphi(\tau) = \tau^s\psi(\tau)$, and the function (5) exists, then for every $\varepsilon > 0$

$$f = g + h,$$

where h has support whose measure is less than ε, and g belongs to $C^{s,\bar{\psi}}$, with

$$\sup_{|\alpha|=s}\sup_{x\neq y} \frac{|D^\alpha(g;x) - D^\alpha(g;y)|}{\bar{\psi}(|x-y|)} = O(\varepsilon^{-1/q})\,V(I_f^k;Q_0).$$

Remark. One can also give an estimate for higher-order differences of the function $D^\alpha g$ on the set $Q_0 \setminus \operatorname{supp} h$.

4. We indicate the connection between $\mathcal{L}_p^k(V_q^\varphi)$ and known function spaces.

A. Sobolev spaces.

Theorem 3. The space $\mathcal{L}_p^k(V_q^\varphi)$, for $\varphi(\tau) = \tau^k$, $1 < q \le p$, and $k > n(q^{-1} - p^{-1})$, is isomorphic to the space $W_q^k \cap L_p$.

Remark. For $k = n(q^{-1} - p^{-1})$ only the proper embedding into $W_q^k \cap L_p$ holds.

For $n = 1$, $p = \infty$, and $k = 1$, we obtain from Theorem 3 the known lemma of F. Riesz (see ([1]), p. 85). From Theorem 1 we now obtain the embedding theorem of S. L. Sobolev ([4]) with a certain refinement in the limiting case (cf. ([5])), and from Theorem 2, a certain new property of functions from Sobolev spaces, namely that, after modification on a set of small measure, they coincide with functions from $C^{k-1,1}$.

B. Lipschitz spaces. In the works ([6–9]) it was shown that if $\varphi(\tau) = \tau^{s+\alpha}$, $0 \le s < k-1$, $0 < \alpha < 1$, or $\varphi(\tau) = \tau^{s+\alpha}$, $s = k-1$, $\alpha = 1$, then $\mathcal{L}_p^k(V_\infty^\varphi)$ is isomorphic to the space $C^{s,\alpha}$. It also follows from the results of ([9]) that if $0 < s \le k-1$ and $\alpha = 0$, then $\mathcal{L}_p^k(V_\infty^\varphi)$ is isomorphic to $C^{s-1,1-0}$, where the space $C^{l,1-0}$ is defined by the condition that the second-

differences of step h of the higher derivatives are majorized by the quantity $O(|h|)$. For $q < \infty$ we have the following result.

Theorem 4. If $\varphi(\tau) = \tau^\alpha$, $0 < \alpha < k$, then for $p \le q$ the continuous embeddings

$$L_p \cap B_q^\alpha \subset \mathcal{L}_p^k(V_q^\varphi) \subset H_p^\alpha,$$

hold, and for $q \leq p$ the embeddings

$$B_p^\alpha \subset \mathcal{L}_p^k(V_q^\varphi) \subset H_q^\alpha \cap L_p.$$

Here H_p^α, B_p^α are the well-known spaces of S. M. Nikol' skii and O. V. Besov (see the survey [10]).

B. **Functions of bounded variation.** For $\varphi(\tau) = \tau^{1/q}$ and $n = 1$, the space $\mathcal{L}_\infty(V^\varphi)$ is isomorphic to the space of functions of bounded q-variation in the sense of Wiener–L. Young. Therefore also for $n > 1$ it is natural to call the functions of the space $\mathcal{L}_\infty(V_q^\varphi)$, where $\varphi(\tau) = \tau^{1/q}$, functions of bounded q-variation. The space $\mathcal{L}_1(V_1^\varphi)$ for $\varphi(\delta) = \tau$ is isomorphic to the space of functions having bounded variation in the sense of Tonelli. More generally, the following holds.

Theorem 5. The space $\mathcal{L}_1^k(V_1^\varphi)$ for $\varphi(\tau) = \tau^k$ is isomorphic to the space BV^k of functions in $L_1(Q_0)$ whose generalized k-derivatives are Borel measures.

Remark. At the same time, the space $\mathcal{L}_1^k(v_1^\varphi)$, $\varphi(\tau) = \tau^k$, is isomorphic to the space W_1^k.

5. One can choose X in $\mathcal{L}_p^k(X)$ also in such a way as to obtain any Lipschitz space in the sense of Calderon (see [11]). In particular, one can obtain the spaces H_p^α of S. M. Nikol' skii or $B_{p,q}^\alpha$ of O. V. Besov.

6. In conclusion we state an interpolation theorem for operators acting from the scale L_p into the space $\mathcal{L}_p^k(X)$.

Let X_0, X_1 be two spaces of functions of the cube $I(Q)$, described in item 2. Denote by X_s, $0 < s < 1$, the space of those functions $I(Q)$ for each of which there are a number λ and functions $I_i \in X_i$, $i = 0, 1$, such that

$$|I(Q)| \leq \lambda |I_0(Q)|^{1-s} |I_1(Q)|^s, \qquad Q \subset Q_0. \tag{6}$$

We take the lower bound of λ in (6), under $\|I_i\|_{X_i} \leq 1$, as the norm in X_s.

Theorem 6. Let a linear operator T act continuously from L_{p_i} to $\mathcal{L}_{p_i}^k(X_i)$, and let its norm be a_i, $i = 0, 1$. Then T acts continuously from L_p to $\mathcal{L}_p^k(X_s)$, where $1/p = (1-s)/p_0 + s/p_1$, and its norm does not exceed $a_0^{1-s} a_1^s$.

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Note: Figure translations are in progress. See original paper for figures.