Reports of the Academy of Sciences of the USSR

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1970. Volume 192, No. 4

UDC 517.9

MATHEMATICS

E. Ya. Roitenberg

ON THE OBSERVABILITY OF SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS IN A HILBERT SPACE

(Presented by Academician A. Yu. Ishlinskii on 25 XI 1969)

The problem of observability has by now been investigated by various methods, and many interesting results have been obtained in it ([1], [2]). In the present paper a new approach to the study of this problem is proposed, making it possible, for a fairly broad class of nonlinear differential equations, to indicate sufficient conditions for observability.

Let E be a Hilbert space and let R be a normed ring of linear bounded operators mapping E into itself. Consider the differential equation

$$dx/dt = A(t)x + \varphi(x) + p(t), \qquad x(t_0) = x_0, \tag{1}$$

where $x = x(t)$, $t_0 \le t < \infty$, is the unknown continuous, continuously differentiable vector-valued function with values in E; $A(t)$, $t_0 \le t < \infty$, is a uniformly bounded and continuous operator-valued function in the sense of the operator norm with values in R; $\varphi(x)$ is a nonlinear vector-valued function with values in E, continuous in E and, for all x and ξ from E, satisfying in E the Lipschitz condition with constant q:

$$\|\varphi(\xi) - \varphi(x)\| \le q\|\xi - x\|; \tag{2}$$

$p(t)$, $t_0 \le t < \infty$, is a bounded and continuous vector-valued function with values in E; x_0 is some constant vector from $S_\rho(\xi_0)$, where $S_\rho(\xi_0)$ is the ball in E of radius ρ with center at the point ξ_0. The vector x_0 is assumed to be unknown.

Introduce for consideration the vector-valued function $y(t)$, $t_0 \le t < \infty$, with values in E:

$$y(t) = C(t)x(t), \tag{3}$$

which we shall call the **trace** of the solution $x(t)$ of equation (1) with initial condition x_0. Here $C(t)$, $t_0 \le t < \infty$, is a uniformly bounded and continuous operator-valued function from R; $x(t)$ is the solution of equation (1), unknown to us, with initial condition x_0. The existence of the operator-valued function $C^{-1}(t)$, $t_0 \le t < \infty$, is not assumed.

For equation (1) and the vector-valued function (3), consider the observability problem.

Definition 1. If, from the trace $y(t)$ of the solution $x(t)$ known on the interval $[t_0, t_1]$, one can determine $x(t_1)$, then we shall say that the solution $x(t)$ of equation (1) is **observable**.

In the case when E is an n-dimensional Euclidean space and system (1) is linear, the problem of finding $x(t_1)$ from a linear combination of the components of the vector $x(t)$ known on the interval $[t_0, t_1]$ was considered by R. Kalman ([1]) and was called by him the observability problem.

Definition 2. If, from the trace $y(t)$ of the solution $x(t)$ of equation (1) known on the interval $[t_0, t_1]$, one can find a continuous, continuously dif-

differentiable vector function $\xi(t)$, $t_0 \le t < \infty$, with values in E, such that, for a prescribed scalar quantity $\mu > 0$, at the instant $t = t_1$ the relation

$$\|\xi(t_1) - x(t_1)\| \le \mu \tag{4}$$

is satisfied.

Then we shall say that the solution $x(t)$ of equation (1) is μ-observable at the instant $t = t_1$.

Definition 3. Suppose that, from the known trace $y(t)$, $t_0 \le t < \infty$, of the solution $x(t)$ of equation (1), one can find a continuous, continuously differentiable vector function $\xi(t)$, $t_0 \le t < \infty$, with values in E, such that, for a prescribed scalar quantity $\mu > 0$, starting from some instant $T(\mu) \in (t_0, \infty)$, for all $t \ge T(\mu)$ the relation

$$\|\xi(t) - x(t)\| \le \mu \tag{5}$$

is satisfied.

Then we shall say that the solution $x(t)$ of equation (1) is asymptotically μ-observable.

It is obvious that if $t_1 \ge T(\mu)$, then inequality (4) follows from inequality (5), and, consequently, asymptotic μ-observability of a solution implies its μ-observability at the instant $t = t_1$.

The vector function $\xi(t)$, $t_0 \leq t < \infty$, will be called a function that realizes observation.

In this paper we shall find conditions sufficient for μ-observability at the instant $t = t_1$ and for asymptotic μ-observability of the solution $x(t)$, and we shall obtain a differential equation whose solution is a function that realizes observation.

Consider the following auxiliary differential equation

$$d\xi/dt = A(t)\xi + \varphi(\xi) + p(t) + u(t), \qquad \xi(t_0) = \xi_0, \tag{6}$$

whose solution $\xi(t)$ is assumed known to us. Here $\xi(t)$, $t_0 \leq t < \infty$, is a continuous, continuously differentiable vector function with values in E; $u(t)$, $t_0 \leq t < \infty$, is some vector function with values in E. The trace of the solution of equation (6) with initial condition ξ_0 will be denoted by $\eta(t)$:

$$\eta(t) = C(t)\xi(t). \tag{7}$$

In the space E consider the vector function

$$z = \xi - x. \tag{8}$$

It follows from (1) and (6) that $z(t)$ satisfies the differential equation

$$dz/dt = A(t)z + \varphi(\xi) - \varphi(x) + u(t) \tag{9}$$

with initial condition $z(t_0) = z_0$, where $\|z(t_0)\| \leq \rho$. Denote $\varphi(\xi) - \varphi(x) = F(z, t)$; then equation (9) can be written in the form

$$dz/dt = A(t)z + u(t) + F(z, t), \tag{10}$$

where, as follows from (2), $\|F(z, t)\| \leq q\|z\|$. The corresponding equation of first approximation for equation (10) has the form

$$dz/dt = A(t)z + u(t). \tag{11}$$

We take the vector function $u(t)$ to be the following:

$$u(t) = B(t)(\eta(t) - y(t)), \tag{12}$$

where $B(t)$, $t_0 \leq t < \infty$, is some continuous uniformly bounded operator from R. In accordance with (3), (7), (8), and (12), equation (10) takes the form

$$dz/dt = (A(t) + B(t)C(t))z + F(z,t). \tag{13}$$

Equation (11) is written in the form

$$dz/dt = (A(t) + B(t)C(t))z. \tag{14}$$

Sufficient conditions for the asymptotic μ-observability of the solution $x(t)$ of equation (1) are formulated in the form of the following theorem:

Theorem 1. *For the asymptotic μ-observability of the solution $x(t)$ of equation (1) in the space E, it is sufficient to choose an operator $B(t) = B_V(t) \subset R$ so that there exists a differentiable uniformly positive operator-function $V(t)$*

$$0 < a_1(z,z) \leq (V(t)z,z) \leq a_2(z,z), \tag{15}$$

possessing the property that, if $z(t)$ is a solution of equation (14), then

$$\frac{d}{dt}(V(t)z(t), z(t)) \leq -\beta(z(t), z(t)) \tag{16}$$

and the relation $q_1 = v_0/N_0 > q$ holds, where $v_0 = \beta/2a_2,\ N_0 = \sqrt{a_2/a_1}$.

To prove Theorem 1, we note that, when conditions (15) and (16) are fulfilled, for the norm of the solution $z(t)$ of equation (13) the estimate ([3])

$$\|z(t)\| \leq N_0 e^{-(v_0 - N_0 q)(t-t_0)} \|z(t_0)\|$$

holds for all $\|z(t_0)\| < \rho$.

Then, beginning from the time $T(\mu)$,

$$T(\mu) = [\ln\mu - \ln N_0 - \ln\rho - t_0(v_0 - N_0 q)]/(N_0 q - v_0), \tag{17}$$

for all $t \geq T(\mu)$ inequality (5) will hold, which means asymptotic μ-observability of the solution $x(t)$ of equation (1).

When the conditions of Theorem 1 are fulfilled, there is a theorem that makes it possible to find the function carrying out the observation.

Theorem 2. *The vector-function $\xi(t)$ carrying out asymptotic μ-observation of the solution $x(t)$ of equation (1) is a solution of the differential equation*

$$d\xi/dt = A(t)\xi + \varphi(\xi) + p(t) + B_V(t)C(t)\xi - B_V(t)y(t), \qquad \xi(t_0) = \xi_0.$$

In relation (17), μ, q, and ρ are prescribed scalar quantities; $N_0 = N_0(a_1, a_2)$, $v_0 = v_0(a_2, \beta)$. Obviously, by choosing a_1, a_2, and β, one can ensure that the relation

$$T(\mu) \leq t_1. \tag{18}$$

is fulfilled.

The scalar quantities a_1, a_2, and β for which relation (18) is fulfilled will be denoted by a_1^1, a_2^1, and β^1, respectively. There is a theorem giving sufficient conditions for the μ-observability of the solution $x(t)$ of equation (1) at the time $t = t_1$:

Theorem 3. *For the μ-observability of the solution $x(t)$ of equation (1) at the time $t = t_1$, it is sufficient to choose an operator $B(t) = B_{V_1}(t) \subset R$ so that there exists a bounded operator-function V_1 satisfying relations (15) and (16) for $a_1 = a_1^1$, $a_2 = a_2^1$, and $\beta = \beta^1$.*

When the conditions of Theorem 3 are fulfilled, there is a theorem analogous to Theorem 2:

Theorem 4. *The vector-function $\xi(t)$ carrying out μ-observation of the solution $x(t)$ of equation (1) at the time $t = t_1$ is a solution of the differential equation*

$$d\xi/dt = A(t)\xi + \varphi(\xi) + p(t) + B_{V_1}(t)C(t)\xi - B_{V_1}(t)y, \qquad \xi(t_0) = \xi_0.$$

Corollary 1. *If in the right-hand side of equation (1) there is also an unknown perturbation to us—a vector-function $p(t)$, $t_0 \leq t < \infty$, with values in E, such that $\|p(t)\| < k_1$ for all $t_0 \leq t < \infty$, then the equation*

(13) takes the form

$$dz/dt = (A(t) + B_\nu(t)C(t))z + F(z,t) - p_1(t). \tag{19}$$

For the solution $z(t)$ of equation (19), obviously, the estimate holds

$$\|z(t)\| \leq N_0 e^{-(\nu_0 - N_0 q)(t-t_0)} \|z(t_0)\| + \frac{N_0 k_1}{\nu_0} \left[1 - e^{-\nu_0 (t-t_0)}\right],$$

from which it follows that one can solve the problem of μ-observability under constantly acting perturbations.

Corollary 2. Suppose that relation (3) for the trace $y(t)$ of a solution $x(t)$ of equation (1) has the form

$$y(t) = C(t)x(t) + p_2(t),$$

where $p_2(t)$, $t_0 \leq t < \infty$, is an unknown vector-function with values in E, whose norm $\|p_2(t)\| < k_2$ for all $t_0 \leq t < \infty$. Then equation (13) takes the form

$$dz/dt = (A(t) + B_\nu(t)C(t))z + F(z,t) + p_1(t),$$

where $p_1(t) = B_\nu(t)p_2(t)$ and $\|p_1(t)\| \leq \|B_\nu(t)\| \|p_2(t)\| < b_\nu k_2$, since for all $t_0 \leq t < \infty$, by virtue of the uniform boundedness of the operator-function B_ν, the relation $\|B_\nu(t)\| < b_\nu$ holds. Thus we arrive at the conditions of Corollary 1.

Corollary 3. Suppose that relation (3) for the trace $y(t)$ has the form

$$y(t) = C(t)[x(t) + p_3(t)], \tag{20}$$

where $p_3(t)$, $t_0 \leq t < \infty$, is an unknown vector-function with values in E, $\|p_3(t)\| < k_3$ for all $t_0 \leq t < \infty$. Then, denoting $p_2(t) = C(t)p_3(t)$ and using the fact that $C(t)$ is a uniformly bounded operator-function: $\|C(t)\| < c$ for all $t_0 \leq t < \infty$, we obtain for the norm of $p_2(t)$ the estimate $\|p_2(t)\| = \|C(t)p_3(t)\| < ck_3 = k_2$. Relation (20) takes the form

$$y(t) = C(t)x(t) + p_2(t),$$

and thus we arrive at the conditions of Corollary 2.

Until now, in considering the problem of μ-observability, we have not assumed that the solution $x(t)$ of equation (1) is bounded on the half-interval $t_0 \leq t < \infty$. If we now assume that $\|x(t)\| < k_4$ for all $t_0 \leq t < \infty$, then the following holds.

Corollary 4. Suppose that relation (3) for the trace $y(t)$ of a solution $x(t)$ of equation (1) has the form

$$y(t) = [C(t) + \Phi(t)]x(t), \tag{21}$$

where $C(t)$ is a known operator-function, and $\Phi(t)$, $t_0 \leq t < \infty$, is an unknown uniformly bounded operator-function, $\|\Phi(t)\| < f$. Expression (21) can be written in the form

$$y(t) = C(t)x(t) + p_2(t),$$

where $\|p_2(t)\| = \|\Phi(t)x(t)\| \leq \|\Phi(t)\| \|x(t)\| < fk_4 = k_2$. Thus we find ourselves under the conditions of Corollary 2.

For the finite-dimensional case, sufficient conditions for μ-observability were obtained in the author' s work ([4]). We also note that the method developed above is applicable to finding sufficient conditions for μ-observability of solutions of nonlinear differential equations with retarded argument.

Moscow State University
named after M. V. Lomonosov

Received
13 XI 1969

REFERENCES

1. R. Kalman, Bol. Soc. Mat. Mexicana, 3, No. 1 (1960).

2. N. N. Krasovskii, *Theory of Motion Control*, Moscow, 1968.

3. M. G. Krein, *Lectures on the Theory of Stability of Solutions of Differential Equations in Banach Space*, Kiev, 1964.

4. E. Ya. Roitenberg, Vestn. Mosk. Univ., 1, Mathematics, Mechanics, No. 2 (1969).

Note: Figure translations are in progress. See original paper for figures.